Robust optimization applied to uncertain limit analysis and optimal plastic design

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Limit state design methods in civil engineering

Variability: material properties, loading conditions, boundary conditions, initial state, etc.

Design codes: use of **partial safety coefficients** on loading amplitude and material strength to guard against uncertainty

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limit analysis theory: a tool to estimate a structure collapse load using convex optimization

Outline

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$g_G(\boldsymbol{\sigma}) = \inf\{\lambda > 0 \text{ s.t. } \boldsymbol{\sigma} \in \lambda G\}$ Minkowski functional (gauge) of G

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Collapse load

Find the **maximum** load multiplier λ such that such a stress field exists

Convex optimization formulation : nominal case

Continuous problem:

$$\begin{aligned} \lambda^+ = \max_{\lambda, \boldsymbol{\sigma} \in \mathcal{W}} \quad & \lambda \\ \text{s.t.} \quad & \operatorname{div} \boldsymbol{\sigma} + \lambda \boldsymbol{f} = 0 \qquad \forall \boldsymbol{x} \in \Omega \\ & g_G(\boldsymbol{\sigma}) \leq 1 \end{aligned}$$

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Discrete (e.g. finite-element) formulation:

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convex optimization problems

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convex optimization problems
usually G (thus also g_G) has a simple geometrical shape: ellipsoid, cone, etc.
$\Rightarrow$ **conic programming solvers** e.g. `MOSEK`

Uncertain limit analysis

Uncertain **material strength** and/or **loading**

$$\begin{array}{rll} \lambda^+(\zeta) = \displaystyle\max_{\lambda,\boldsymbol{\sigma}} & \lambda & \\ \text{s.t.} & \boldsymbol{H}\boldsymbol{\sigma} + \lambda\boldsymbol{F}(\zeta) = 0 & \\ & g_{G(\zeta)}(\boldsymbol{\sigma}_k) \leq 1 & \forall k = 1, \dots, N \end{array}$$

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Previous works:

- Monte-Carlo [Huang et al., 2013; Kasama and Whittle, 2016; Ali, 2016]
- mixed integer programming [Kanno and Takewaki, 2007]
- info-gap decision theory [Matsuda and Kanno, 2008]
- FORM/SORM [Staat, 2014]
- chance-constrained programming [Tran et al., 2018; Tran and Staat, 2020, 2021]

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Robust optimization formulation [Ben Tal et al., 2009; Bertsimas et al., 2011]

$\mathcal{U}$ a given **uncertainty** set. Find the worst-case limit load:

$$\lambda_{\text{wc}} = \min_{\zeta\in\mathcal{U}} \max_{\lambda,\boldsymbol{\sigma}} \quad \lambda$$
$$\text{s.t.} \quad \begin{aligned} &\boldsymbol{H\sigma} + \lambda \boldsymbol{F}(\zeta) = 0 \\ &g_{G(\zeta)}(\boldsymbol{\sigma}_k) \leq 1 \qquad \forall k = 1, \ldots, N \end{aligned}$$

In general, $\lambda(\zeta)$ and $\boldsymbol{\sigma}(\zeta)$ are *adjustable variables* ⇒ not tractable in general

Outline

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Strength uncertainty: small in amplitude, assume that (*conservative*):

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$$\begin{aligned} \lambda_{\text{RC}} = \max_{\lambda,\boldsymbol{\sigma}} \min_{\boldsymbol{\zeta}\in\mathcal{U}} \quad & \lambda \\ \text{s.t.} \quad & \boldsymbol{H}\boldsymbol{\sigma} + \lambda \boldsymbol{F} = 0 \\ & g_{G(\zeta)}(\boldsymbol{\sigma}_k) \leq 1 \qquad \forall \boldsymbol{\zeta} \in \mathcal{U},\ \forall k = 1, \dots, N \end{aligned} \tag{RC}$$

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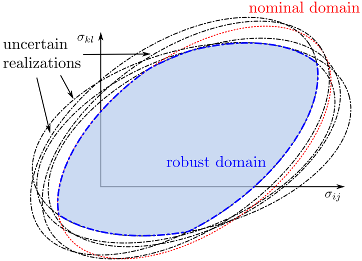
we can replace the uncertain strength condition with a **deterministic** condition:

$$\begin{aligned} \Leftrightarrow \quad & g_{G(\boldsymbol{\zeta})}(\boldsymbol{\sigma}_k) \leq 1 \quad \forall \boldsymbol{\zeta} \in \mathcal{U} \\ \Leftrightarrow \quad & \boldsymbol{\sigma}_k \in G(\boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \mathcal{U} \\ & \boldsymbol{\sigma}_k \in G_{\text{RC}} := \bigcap_{\boldsymbol{\zeta}\in\mathcal{U}} G(\boldsymbol{\zeta}) \end{aligned}$$



Robust Mohr-Coulomb criterion

uncertain **cohesion** $c(\zeta)$ and **friction angle** $\phi(\zeta)$:

$$\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi(\zeta) \leq 2c(\zeta)\cos\phi(\zeta)$$

Uncertainty modelling: assume $\boldsymbol{k} = (c, \phi)$

$$\boldsymbol{k}(\zeta) = \boldsymbol{k}_0 + \boldsymbol{K}\zeta \quad \text{with } \boldsymbol{K} = \begin{bmatrix} \Delta c & \rho\Delta\phi\frac{c_0}{\phi_0} \\ 0 & \sqrt{1-\rho^2}\Delta\phi \end{bmatrix} \text{ and } \zeta \in \mathcal{U}$$

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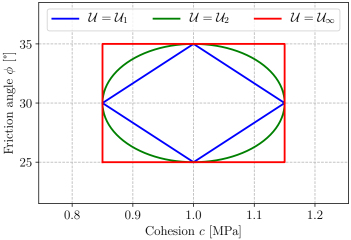
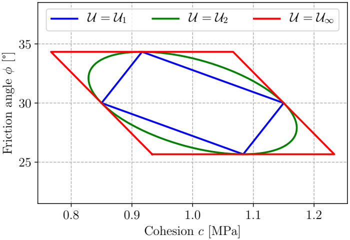
Example: $c_0 = 1$ MPa, $\phi_0 = 30°$, $\Delta c = 150$ kPa, $\Delta\phi = 5°$, $\mathcal{U} = L_p$-ball



(a) $\rho = 0$



(b) $\rho = -0.5$

Tractability of robust strength condition

Tractability of (RC) if one has a tractable conic formulation for G_{RC}
For Mohr-Coulomb, we have:

$$\begin{aligned} g_{\mathrm{MC}}(\boldsymbol{\sigma};\zeta) &= g(\boldsymbol{A}(\zeta)\boldsymbol{\sigma}), \quad \forall \zeta \in \mathcal{U} \\ &\approx g((\boldsymbol{A}_0 + \boldsymbol{A}'_0\zeta)\boldsymbol{\sigma}) \leq 1, \quad \forall \zeta \in \mathcal{U} \end{aligned}$$

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A generic question

Tractable strength constraint reformulation of:

$$g(\boldsymbol{\sigma}+\boldsymbol{\Sigma}\zeta) \leq 1, \quad \forall\zeta\in\mathcal{U} \qquad (1)$$

for a known $\boldsymbol{\Sigma}$? (note: convex in both $\boldsymbol{\sigma}$ and ζ)

Exact reformulations

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- G is **polyhedral** : $G = \{\boldsymbol{\sigma} \text{ s.t. } \boldsymbol{c}_k^\mathsf{T}\boldsymbol{\sigma} \leq d_k, \quad \forall k = 1, \dots, K\}$

$$\begin{aligned} & \boldsymbol{c}_k^\mathsf{T}\boldsymbol{\sigma} + \boldsymbol{c}_k^\mathsf{T}\boldsymbol{\Sigma}\zeta \leq d_k \quad \forall \zeta \in \mathcal{U} \\ \Leftrightarrow \quad & \boldsymbol{c}_k^\mathsf{T}\boldsymbol{\sigma} + \underbrace{\max_{\zeta \in \mathcal{U}}\{(\boldsymbol{c}_k^\mathsf{T}\boldsymbol{\Sigma})\zeta\}}_{:=\|\boldsymbol{\Sigma}^\mathsf{T}\boldsymbol{c}_k\|_{\mathcal{U},*}} \leq d_k \end{aligned}$$

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- $\mathcal{U}$ is **polyhedral** : $\mathcal{U} = \text{conv}\{\zeta_1, \dots, \zeta_N\}$ of vertices ζ_k:

$$\begin{aligned} & g(\boldsymbol{\sigma} + \boldsymbol{\Sigma}\zeta) \leq 1, \quad \forall \zeta \in \mathcal{U} \\ \Leftrightarrow \quad & g(\boldsymbol{\sigma} + \boldsymbol{\Sigma}\zeta_k) \leq 1, \quad \forall k = 1, \dots, N \end{aligned}$$

impractical problem size in general

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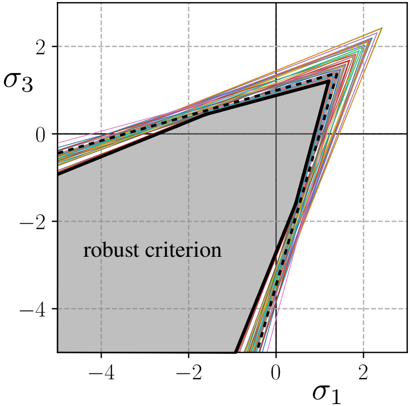
- [Roos et al., 2018] : for a **polyhedral uncertainty** set $\mathcal{U} = \{\boldsymbol{\zeta} \in \mathbb{R}^m \text{ s.t. } \exists \boldsymbol{\xi} \in \mathbb{R}^q \text{ and } \mathrm{D}_1\boldsymbol{\zeta} + \mathrm{D}_2\boldsymbol{\xi} \leq \mathrm{d}\}$ with $\mathrm{D}_1 \in \mathbb{R}^{r\times m}, \mathrm{D}_2 \in \mathbb{R}^{r\times q}$, $d \in \mathbb{R}^r$

$$\exists \boldsymbol{v} \in \mathbb{R}^r,\ \boldsymbol{V} \in \mathbb{R}^{d\times r} \text{ s.t. } \begin{cases} g\left(\boldsymbol{\sigma} - \boldsymbol{V}\boldsymbol{d}\right) + \boldsymbol{d}^\mathsf{T}\boldsymbol{v} \leq 1 \\ g(\boldsymbol{V}_i) \leq v_i \quad \forall i = 1, \ldots, r \\ \boldsymbol{D}_1^\mathsf{T}\boldsymbol{v} = \boldsymbol{b} \\ \boldsymbol{D}_2^\mathsf{T}\boldsymbol{v} = 0 \\ \boldsymbol{V}\boldsymbol{D}_1 + \boldsymbol{\Sigma} = 0 \\ \boldsymbol{V}\boldsymbol{D}_2 = 0 \end{cases} \qquad (3)$$

  is a safe approximation of (1). It is tighter than (2).

Robust Mohr-Coulomb criterion

$\rho = 0$, $\mathcal{U} = \mathcal{U}_\infty$



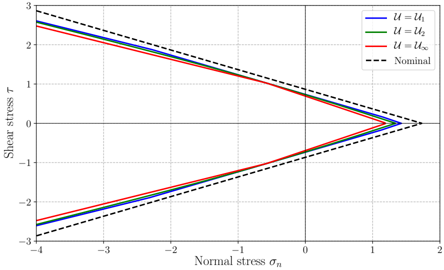
robust criterion *approximately* given by

$$\begin{cases} \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi_{\text{max}} \leq 2c_{\text{min}}\cos\phi_{\text{max}} \\ \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi_{\text{min}} \leq 2c_{\text{min}}\cos\phi_{\text{min}} \end{cases}$$

where $\phi_{\text{max/min}} = \phi_0 \pm \Delta\phi$, $c_{\text{min}} = c_0 - \Delta c$

Robust Mohr-Coulomb criterion

Influence of $\mathcal{U}$ ($\rho = 0$):



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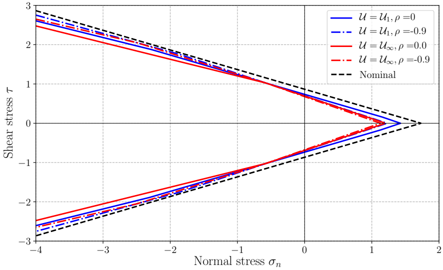
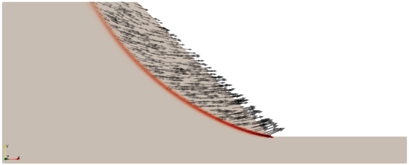


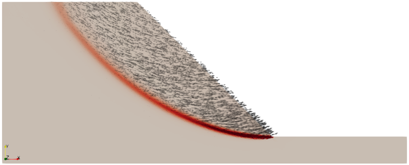
Robust slope stability

standard FELA with robust Mohr-Coulomb criterion (`FEniCS` + `MOSEK`)[1]

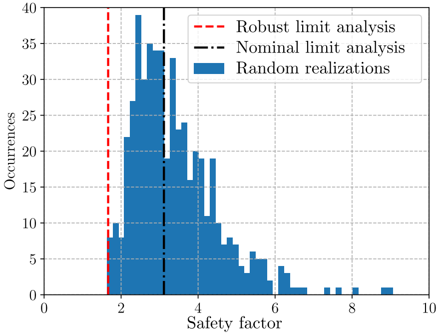
Nominal:



Robust:



Limit load estimates

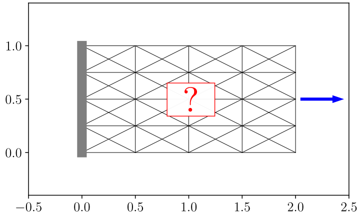


[1] `fenics_optim` open-source package

Outline

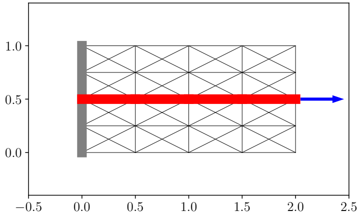
Plastic truss optimization

$$\begin{aligned}\min_{\boldsymbol{a},\boldsymbol{N}} \quad & \ell^{\mathsf{T}}\boldsymbol{a} \\ \text{s.t.} \quad & \boldsymbol{H}\boldsymbol{N} = \boldsymbol{F} \\ & |\boldsymbol{N}| \leq \boldsymbol{a}\end{aligned}$$



Plastic truss optimization

Problem: solutions are so optimized that they are not **robust**



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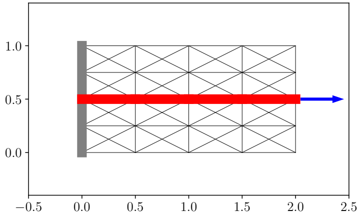

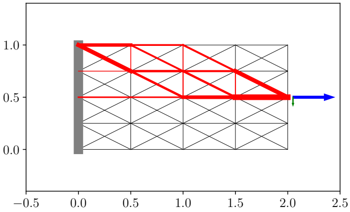

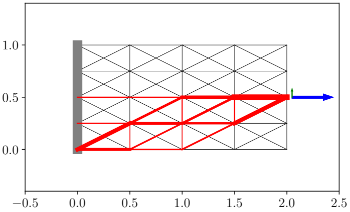

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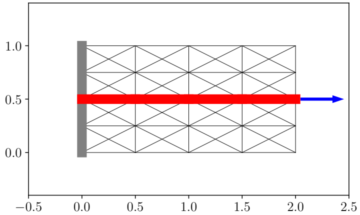

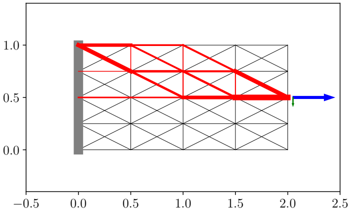

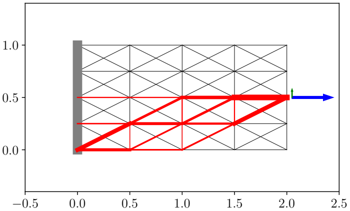

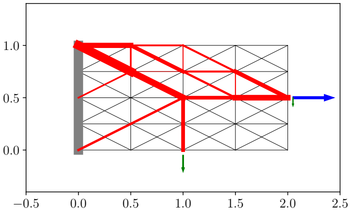

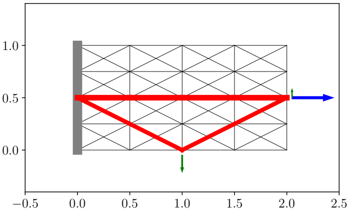

Uncertain plastic design

known, **uncertain**, **optimization variable**:

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Uncertainty set: $\mathcal{U} = \{\zeta \in \mathbb{R}^m \text{ s.t. } \|\zeta\| \leq 1\}$

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$$\boldsymbol{F} = \boldsymbol{F}(\zeta) := \boldsymbol{F}_0 + \sum_i \boldsymbol{F}_i \zeta_i$$

optimize the **worst-case**: (*robust counterpart*)

$$\begin{aligned} \min_{\boldsymbol{a},\boldsymbol{N}} \max_{\zeta\in\mathcal{U}} \quad & \boldsymbol{\ell}^\mathsf{T}\boldsymbol{a} \\ \text{s.t.} \quad & \boldsymbol{H}\boldsymbol{N} = \boldsymbol{F}(\zeta) \\ & |\boldsymbol{N}| \leq \boldsymbol{a} \end{aligned} \iff \begin{aligned} \min_{\boldsymbol{a},\boldsymbol{N}} \quad & \boldsymbol{\ell}^\mathsf{T}\boldsymbol{a} \\ \text{s.t.} \quad & \boldsymbol{H}\boldsymbol{N} = \boldsymbol{F}(\zeta) \quad \forall \zeta \in \mathcal{U} \\ & |\boldsymbol{N}| \leq \boldsymbol{a} \end{aligned}$$

Uncertain plastic design

known, **uncertain**, **optimization variable**:

$$\begin{aligned} \min_{\boldsymbol{a},\boldsymbol{N}} \quad & \boldsymbol{\ell}^\mathsf{T}\boldsymbol{a} \\ \text{s.t.} \quad & \boldsymbol{H}\boldsymbol{N} = \boldsymbol{F} \\ & |\boldsymbol{N}| \leq \boldsymbol{a} \end{aligned}$$

Uncertainty set: $\mathcal{U} = \{\zeta \in \mathbb{R}^m \text{ s.t. } \|\zeta\| \leq 1\}$

$$\boldsymbol{F} = \boldsymbol{F}(\zeta) := \boldsymbol{F}_0 + \sum_i \boldsymbol{F}_i \zeta_i$$

optimize the **worst-case**: (*robust counterpart*)

$$\begin{aligned} \min_{\boldsymbol{a},\boldsymbol{N}} \max_{\zeta\in\mathcal{U}} \quad & \boldsymbol{\ell}^\mathsf{T}\boldsymbol{a} \\ \text{s.t.} \quad & \boldsymbol{H}\boldsymbol{N} = \boldsymbol{F}(\zeta) \\ & |\boldsymbol{N}| \leq \boldsymbol{a} \end{aligned} \iff \begin{aligned} \min_{\boldsymbol{a},\boldsymbol{N}} \quad & \boldsymbol{\ell}^\mathsf{T}\boldsymbol{a} \\ \text{s.t.} \quad & \boldsymbol{H}\boldsymbol{N} = \boldsymbol{F}(\zeta) \quad \forall \zeta \in \mathcal{U} \\ & |\boldsymbol{N}| \leq \boldsymbol{a} \end{aligned}$$

impossible in general to find a single $\boldsymbol{N}$ in equilibrium with all $\boldsymbol{F}(\zeta)$

Adjustable robust optimization

Fixed decision variables: $\boldsymbol{a}$ does not depend on uncertainty
Adjustable (recourse) variables: $\boldsymbol{N}(\zeta)$ should adapt to uncertainty

$$\begin{aligned}\min_{\boldsymbol{a},\boldsymbol{N}(\zeta)} \quad & \ell^\mathsf{T}\boldsymbol{a} \\ \text{s.t.} \quad & \boldsymbol{H}\boldsymbol{N}(\zeta) = \boldsymbol{F}(\zeta) \quad \forall \zeta \in \mathcal{U} \\ & |\boldsymbol{N}(\zeta)| \leq \boldsymbol{a}\end{aligned}$$

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Fixed decision variables: $\boldsymbol{a}$ — does not depend on uncertainty
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NP-hard problem $\Rightarrow$ we restrict to an **affine** recourse $\boldsymbol{N}(\zeta) = \boldsymbol{N}_0 + \sum_{i=1}^m \boldsymbol{N}_i \zeta_i$

$$\begin{aligned} \min_{\boldsymbol{a},\boldsymbol{N}_i} \quad & \boldsymbol{\ell}^\mathsf{T}\boldsymbol{a} \\ \text{s.t.} \quad & \boldsymbol{H}\boldsymbol{N}_i = \boldsymbol{F}_i \quad && \forall i = 0, \dots, m \\ & |\boldsymbol{N}_0 + \sum_{i=1}^m \boldsymbol{N}_i \zeta_i| \leq \boldsymbol{a} && \forall \zeta \in \mathcal{U} \end{aligned}$$

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NP-hard problem $\Rightarrow$ we restrict to an **affine** recourse $\boldsymbol{N}(\zeta) = \boldsymbol{N}_0 + \sum_{i=1}^{m} \boldsymbol{N}_i \zeta_i$

$$\begin{aligned} \min_{\boldsymbol{a},\boldsymbol{N}_i} \quad & \ell^\mathsf{T}\boldsymbol{a} \\ \text{s.t.} \quad & \boldsymbol{H}\boldsymbol{N}_i = \boldsymbol{F}_i \qquad \forall i = 0, \dots, m \\ & \max_{\zeta \in \mathcal{U}} \{|\boldsymbol{N}_0 + \sum_{i=1}^{m} \boldsymbol{N}_i \zeta_i|\} \leq \boldsymbol{a} \end{aligned}$$

maximize a convex function is **NP-hard** in general

Adjustable robust optimization

Fixed decision variables: $\boldsymbol{a}$ — does not depend on uncertainty
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$$\begin{array}{rll} \displaystyle\min_{\boldsymbol{a},\boldsymbol{N}(\zeta)} & \ell^\mathsf{T}\boldsymbol{a} & \\ \text{s.t.} & \boldsymbol{H}\boldsymbol{N}(\zeta) = \boldsymbol{F}(\zeta) & \forall \zeta \in \mathcal{U} \\ & |\boldsymbol{N}(\zeta)| \leq \boldsymbol{a} & \end{array}$$

NP-hard problem $\Rightarrow$ we restrict to an **affine** recourse $\boldsymbol{N}(\zeta) = \boldsymbol{N}_0 + \sum_{i=1}^{m} \boldsymbol{N}_i \zeta_i$

$$\begin{array}{rll} \displaystyle\min_{\boldsymbol{a},\boldsymbol{N}_i} & \ell^\mathsf{T}\boldsymbol{a} & \\ \text{s.t.} & \boldsymbol{H}\boldsymbol{N}_i = \boldsymbol{F}_i & \forall i = 0, \dots, m \\ & \displaystyle\max_{\zeta\in\mathcal{U}} \{\boldsymbol{N}_0 + \sum_{i=1}^{m} \boldsymbol{N}_i \zeta_i\} \leq \boldsymbol{a} & \\ & \displaystyle\max_{\zeta\in\mathcal{U}} \{-\boldsymbol{N}_0 - \sum_{i=1}^{m} \boldsymbol{N}_i \zeta_i\} \leq \boldsymbol{a} & \end{array}$$

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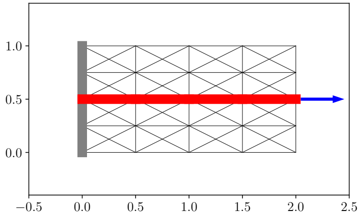
$$\begin{array}{rll} \displaystyle\min_{\boldsymbol{a},\boldsymbol{N}_i} & \ell^{\mathsf{T}}\boldsymbol{a} & \\ \text{s.t.} & \boldsymbol{H}\boldsymbol{N}_i = \boldsymbol{F}_i & \forall i = 0, \dots, m \\ & \boldsymbol{N}_0 + \|(\boldsymbol{N}_i)\|_* \leq \boldsymbol{a} & \\ & -\boldsymbol{N}_0 + \|(-\boldsymbol{N}_i)\|_* \leq \boldsymbol{a} & \end{array}$$

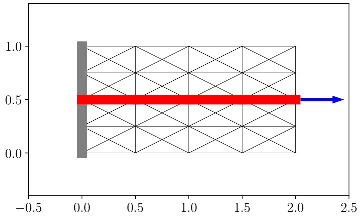
convex problem of size $((2+m)N_{\text{el}})$ vs. $2N_{\text{el}}$ in the nominal case

Results

Case with 1 uncertain loading:



Case with 2 uncertain loadings:



Conclusions and perspectives

Robust limit analysis theory

- RO concepts are applied to convex optimization problems of limit analysis
- may require affine decision rules (e.g. load uncertainty)
- robust strength constraint need specific approximate reformulations
- resulting RC remains tractable and computationally efficient

Bleyer, J., & Leclère, V. (2022). `https://arxiv.org/abs/2203.11354`

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Perspectives

- upper bound kinematic formulation, numerical implementation
- shakedown analysis
- probabilistic guarantees ? non-linear decision rules ?
- applications : masonry, frame structures, geotechnics etc.

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Thank you for your attention !

Particular case of box-uncertainty

[Roos et al. approximation] in the **box-uncertainty** case $\mathcal{U} = \mathcal{U}_\infty$:

$$g(\boldsymbol{\sigma} + \boldsymbol{\Sigma}\zeta) \leq 1, \quad \forall \boldsymbol{\zeta} \in \mathcal{U}_\infty$$

is safely approximated by $\exists \boldsymbol{w} \in \mathbb{R}^m,\ \boldsymbol{W} \in \mathbb{R}^{d\times m}$ s.t.

$$\begin{cases} \displaystyle\sum_{j=1}^m w_j + g\left(\boldsymbol{\sigma} - \sum_{j=1}^m \boldsymbol{W}_j\right) \leq 1 \\ \max\{g(\boldsymbol{W}_j - \boldsymbol{\Sigma}_j); g(\boldsymbol{W}_j + \boldsymbol{\Sigma}_j)\} \leq w_j \quad \forall j = 1, \dots, m \end{cases}$$