Stochastic formulation of generalized standard materials

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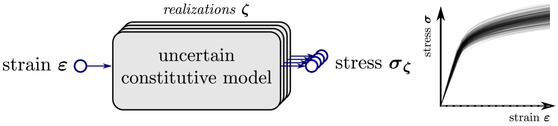
Objectives

Material constitutive law: $\boxed{\boldsymbol{\sigma} = \mathsf{F}(\boldsymbol{\varepsilon})}$

uncertain material properties $\Rightarrow$ need for an **effective behavior**



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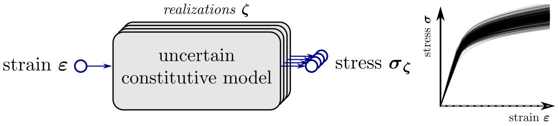

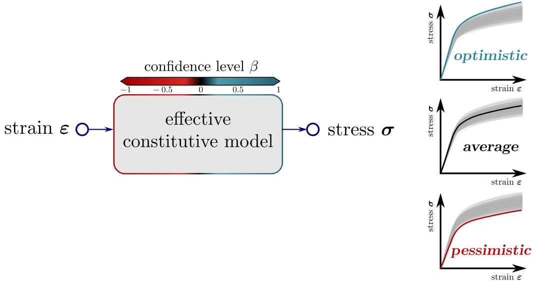

must account for **history-dependent** behaviors and **thermodynamic consistency**

Generalized standard materials

Dissipative materials can be modeled using the GSM framework [Halphen & Nguyen, 1983]

- state variables: $\boldsymbol{\varepsilon}, \boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = p, \boldsymbol{\varepsilon}^{\text{p}}, d, f$, etc.
- free energy: $\psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})$
- pseudo-dissipation potential: $\phi(\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\alpha}})$

ψ and ϕ are **convex**, **non-negative** and **zero at the origin**

Evolution equations

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{nd}} + \boldsymbol{\sigma}^{\text{d}}$$
$$0 = \boldsymbol{Y}^{\text{nd}} + \boldsymbol{Y}^{\text{d}}$$
$$(\boldsymbol{\sigma}^{\text{nd}}, \boldsymbol{Y}^{\text{nd}}) \in \partial_{(\boldsymbol{\varepsilon},\boldsymbol{\alpha})}\psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})$$
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Advantages

- satisfies thermodynamic requirements such as **positive dissipation**, **stability**
- handles **rate-dependent** and **rate-independent** behaviors (for $\phi(\boldsymbol{x})$ homogeneous of degree 1)
- $\phi(\boldsymbol{x})$ usually **non-smooth** (plasticity)

Evolution equations

After time discretization, evolution equations are obtained from the incremental potential [Ortiz & Stainier, Mielke, etc.]

$$\psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) + \Delta t \phi\left(\frac{\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_n}{\Delta t}, \frac{\boldsymbol{\alpha}-\boldsymbol{\alpha}_n}{\Delta t}\right)$$

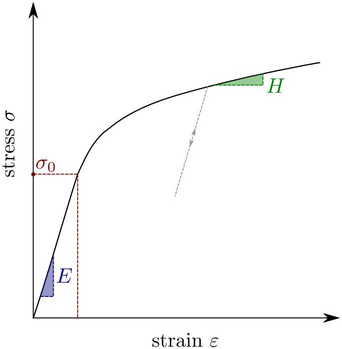
Simplifying assumptions: $\boldsymbol{\varepsilon}$ is non-dissipative, ϕ is 1-homogeneous, single-step $\boldsymbol{\alpha}_n = 0$

$$j(\boldsymbol{\varepsilon}) := \inf_{\alpha} \psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) + \phi(\boldsymbol{\alpha})$$

$$\Rightarrow \boldsymbol{\sigma} \in \partial_{\boldsymbol{\varepsilon}} j$$

Example: 1D linear elasticity + isotropic power-law hardening

$$\begin{aligned}\psi(\varepsilon,\alpha) &= \psi_{\mathsf{el}}(\varepsilon-\alpha) + \psi_{\mathsf{h}}(\alpha) \\ &= \frac{1}{2}E(\varepsilon-\alpha)^2 + \frac{1}{m}H\alpha^m \\ \phi(\dot{\alpha}) &= \sigma_0|\dot{\alpha}|\end{aligned}$$



Uncertain case

Now j depends upon **stochastic parameters** ζ with known probability distribution
Goal: formulate an **effective potential** to describe the effective behavior

$$\boxed{j^{\text{eff}}(\varepsilon) = \mathcal{R}\left[j(\varepsilon;\zeta)\right]}$$

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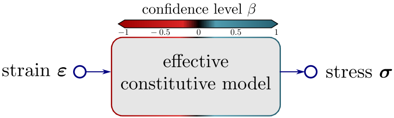
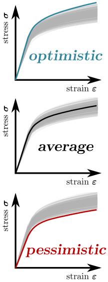
$$\psi(\varepsilon,\alpha) = \frac{1}{2}E_\zeta(\varepsilon-\alpha)^2 + \frac{1}{m}H_\zeta\alpha^m \quad ; \quad \phi(\dot{\alpha}) = \sigma_{0\zeta}|\dot{\alpha}|$$





Outline

Average behavior

Uncertain convex potential: $j(\varepsilon;\zeta)$, conjugate potential $j^*(\boldsymbol{\sigma};\zeta)$

Stochastic programming framework

Two possibilities:

1. ε is a **first-stage** variable: $j^{\text{eff}}(\varepsilon) = \mathbb{E}\left[j\right](\varepsilon)$
2. $\boldsymbol{\sigma}$ is a **first-stage** variable $j^{*,\text{eff}}(\boldsymbol{\sigma}) = \mathbb{E}\left[j^*\right](\boldsymbol{\sigma})$

(internal state variables α are always **second-stage** variables)

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e.g. **Elasticity**: $j(\boldsymbol{\varepsilon}_\zeta;\zeta) = \dfrac{1}{2}\boldsymbol{\varepsilon}_\zeta : \mathbb{C}_\zeta : \boldsymbol{\varepsilon}_\zeta$

$$j^{\text{eff}}(\boldsymbol{\varepsilon}) = \mathbb{E}\left[\frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}_\zeta : \boldsymbol{\varepsilon}\right] = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{E}\left[\mathbb{C}_\zeta\right] : \boldsymbol{\varepsilon}$$

or

$$j^{\text{eff}}(\boldsymbol{\varepsilon}) = \begin{array}{rl} \displaystyle\inf_{\boldsymbol{e}_\zeta} & \mathbb{E}\left[\dfrac{1}{2}\boldsymbol{\varepsilon}_\zeta : \mathbb{C}_\zeta : \boldsymbol{\varepsilon}_\zeta\right] = \dfrac{1}{2}\boldsymbol{\varepsilon} : \mathbb{E}\left[\mathbb{C}_\zeta^{-1}\right]^{-1} : \boldsymbol{\varepsilon} \\ \text{s.t.} & \mathbb{E}\left[\boldsymbol{\varepsilon}_\zeta\right] = \boldsymbol{\varepsilon} \end{array}$$

Uncertain elastoplasticity with non-linear hardening

$$j(\varepsilon_\zeta, \alpha_\zeta) = \frac{1}{2}E_\zeta(\varepsilon_\zeta - \alpha_\zeta)^2 + \frac{1}{m}H_\zeta(\alpha_\zeta)^m + \sigma_{0\zeta}|\alpha_\zeta|$$

Primal formulation: $\boldsymbol{\varepsilon}$ as first-stage

$$j^{\text{eff}}(\varepsilon) = \inf_{\alpha_\zeta} \quad \frac{1}{2}\mathbb{E}\left[E_\zeta(\varepsilon - \alpha_\zeta)^2\right] + \frac{1}{m}\mathbb{E}\left[H_\zeta(\alpha_\zeta)^m\right] + \mathbb{E}\left[\sigma_{0\zeta}|\alpha_\zeta|\right]$$

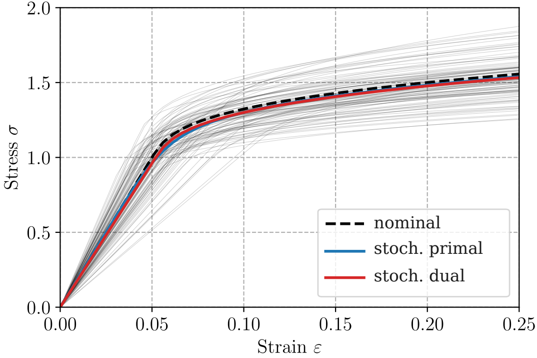
Dual formulation: $\boldsymbol{\sigma}$ as first-stage

$$\begin{aligned} j^{\text{eff}}(\varepsilon) = \inf_{e_\zeta, \alpha_\zeta} \quad & \frac{1}{2}\mathbb{E}\left[E_\zeta(\varepsilon + e_\zeta - \alpha_\zeta)^2\right] + \frac{1}{m}\mathbb{E}\left[H_\zeta(\alpha_\zeta)^m\right] \\ & + \sup_\zeta[\sigma_0(\zeta)|\alpha_\zeta|] \\ \text{s.t.} \quad & \mathbb{E}\left[e_\zeta\right] = 0 \end{aligned}$$

$\Rightarrow$ note how free energy and dissipation are **treated differently!** $\mathbb{E}$ vs sup

Numerical results

Monte-Carlo sampling approximation: GSM with N internal variables solved using `cvxpy`, 30 load steps



Hardening case $\bar{H} = \bar{E}/20$

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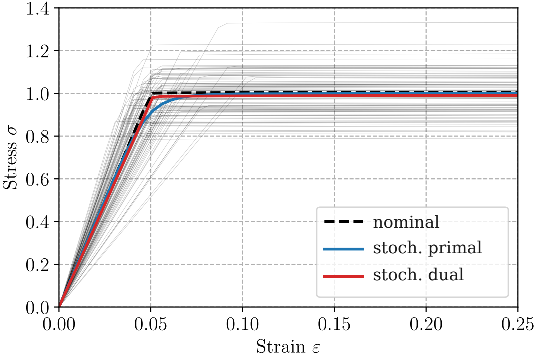

Nearly perfectly plastic case $\bar{H} = \bar{E}/2000$

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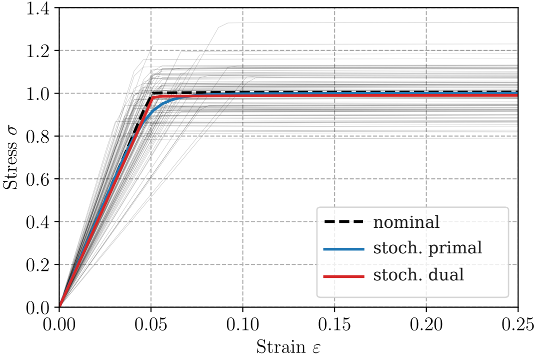

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some differences: **progressive plasticity** onset ("structural hardening"), **almost similar yield strength** $\mathbb{E}[X] \approx \mathbb{E}\left[X^{-1}\right]^{-1}$ for lognormal variables with small variance

Outline

Risk-averse measures

Risk-measure $\mathcal{R}$: assume X be a **random cost**: $\mathbb{E}[X] =$ OK whereas $\mathcal{R}[X] =$ BAD e.g.:

- the safety margin $\mathcal{R}[X] = \mathbb{E}[X] + k\,\text{std}[X], \quad \text{for } k > 0$
- the worst-case value: $\mathcal{R}[X] = \sup X$
- the *Value-at-Risk* (VaR) for a level $\beta \in [0;1]$ (or the β-quantile):

$$\mathcal{R}[X] = \text{VaR}_\beta(X) = \inf\{Z \text{ s.t. } \mathbb{P}[Z \geq X] \geq \beta\}$$

- and many more...

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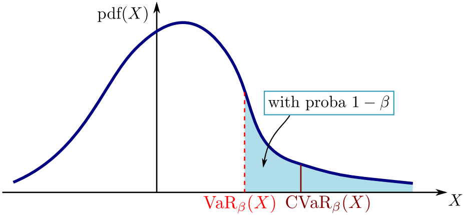
we look for **good mathematical properties** such as **convexity**, **monotonicity**, **homogeneity**
$\Rightarrow$ **coherent risk measures** [Artzner, 1999]

- **safety margin** and **VaR** are **not coherent**
- **worst-case value** is coherent but **too conservative**
- **expected value** is coherent but **risk-neutral**

Conditional Value-at-Risk (CVaR)

The **Conditional Value-at-Risk** (CVaR) is a **coherent risk measure** [Rockafellar, 2000]

$$\mathsf{CVaR}_\beta(X) = \mathbb{E}[X \text{ s.t. } X \geq \mathsf{VaR}_\beta(X)]$$



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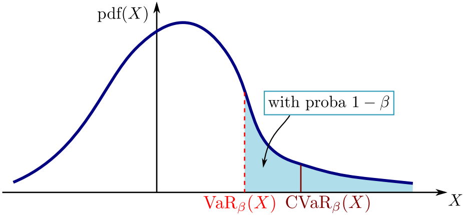

Key result: **convex optimization characterization**

$$\mathsf{CVaR}_\beta(X) = \inf_\lambda \lambda + \frac{1}{1-\beta}\mathbb{E}\left[\langle X-\lambda\rangle_+\right]$$

Extends to **random convex functions**:

$$\boxed{\mathsf{CVaR}_\beta(f)(\boldsymbol{x}) = \inf_\lambda \lambda + \frac{1}{1-\beta}\mathbb{E}\left[\langle f(\boldsymbol{x};\zeta)-\lambda\rangle_+\right]} \text{ **is convex**}$$

Examples

1D elasticity: $j(\varepsilon_\zeta;\zeta) = \dfrac{1}{2}E_\zeta\varepsilon_\zeta^2$

$$j^{\text{eff}}(\varepsilon) = \text{CVaR}_\beta\left(\psi\right)(\varepsilon) = \frac{1}{2}\text{CVaR}_\beta\left(E\right)\varepsilon^2$$

replaces uncertain Young modulus E_ζ with an **optimistic estimate** $\text{CVaR}_\beta\left(E\right)$

General case

take **CVaR** on **free-energy** and **dissipation potential** + $\boldsymbol{\varepsilon}$ first-stage

$$j^{\text{eff}}(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{\alpha}_\zeta}\text{CVaR}_\beta\left(\psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}_\zeta;\boldsymbol{\zeta})\right) + \text{CVaR}_\beta\left(\phi(\boldsymbol{\alpha}_\zeta;\boldsymbol{\zeta})\right)$$

if $\beta = 0$, $\text{CVaR} = \mathbb{E}$ and we recover the **primal formulation** for the average behavior $\Rightarrow$ results in **optimistic** stiffness, strength and hardening

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what about **pessimistic estimates** ?

Dual CVaR

need for a "left-tail" CVaR which still yields a **convex potential**
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$$j^{\text{eff}}(\varepsilon) = \text{dCVaR}_\beta(j)(\varepsilon) = \frac{1}{2}\,\text{CVaR}_\beta\left(E^{-1}\right)^{-1}\varepsilon^2$$

seems OK: $\text{dCVaR}_0(E) = \mathbb{E}\left[E^{-1}\right]^{-1}$ and $\text{dCVaR}_1(E) = \sup\{E^{-1}\}^{-1}$

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But... not coherent, lacks **homogeneity**
$\Rightarrow$ for a plastic potential $\phi(\dot{\alpha}) = \sigma_{0\zeta}\dot{\alpha}$, we show that

$$\mathrm{dCVaR}_\beta(\phi) = \inf_\zeta \sigma_{0\zeta}\dot{\alpha}$$

too pessimistic, irrespective of the confidence level β.

Correct dual CVaR

$$\boxed{\mathrm{dCVaR}_\beta\left(j\right) = \mathrm{CVaR}_\beta\left(j^\circ\right)^\circ}$$

where $f^\circ(\boldsymbol{x}) = \inf\{\mu \geq 0 \text{ s.t. } \mu f^*(\boldsymbol{x}/\mu) \leq 1\}$ is the **polar** of f [Rockafellar, 1970]

Properties:

- $f^\circ \geq g^\circ$ if $f \leq g$
- $j^\circ = j^*$ if j quadratic
- $\mathrm{dCVaR}_\beta\left(\phi\right)\left(\dot{\alpha}\right) = \mathrm{CVaR}_\beta\left(\sigma_{\mathbf{0}}^{-\mathbf{1}}\right)^{-\mathbf{1}}|\dot{\alpha}|$ for the plastic potential
- $\mathcal{R}[j] = \mathrm{dCVaR}_\beta\left(j\right)$ is a **coherent risk measure**

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Convex representation

$$\begin{aligned}\mathrm{dCVaR}_\beta(f)(\boldsymbol{x}) = \inf_{v\geq 0,\widehat{\boldsymbol{x}}} \quad & \max\left\{\mathbb{E}\left[vf(\widehat{\boldsymbol{x}}/v;\zeta)\right]; (1-\beta)\sup_{\zeta}\{vf(\widehat{\boldsymbol{x}}/v;\zeta)\}\right\} \\ \text{s.t.} \quad & \mathbb{E}\left[\widehat{\boldsymbol{x}}\right] = \boldsymbol{x} \\ & \mathbb{E}\left[v\right] = 1\end{aligned}$$

Risk-averse behavior of stochastic GSM

Optimistic estimate: take **CVaR** of **free-energy** and **dissipation potential** + $\boldsymbol{\varepsilon}$ as first-stage

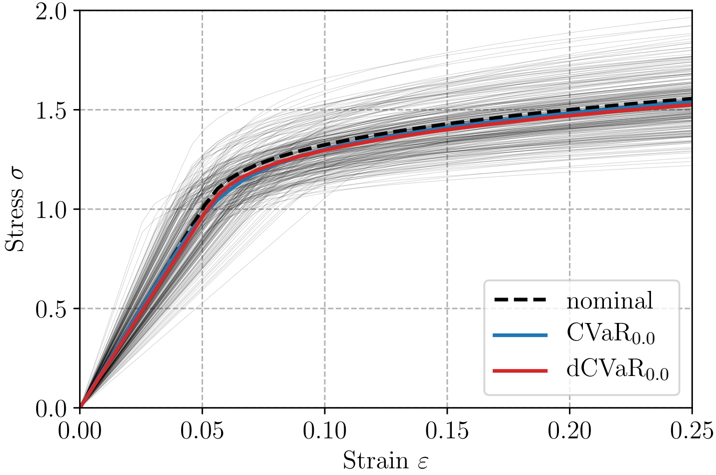
$$j^{\text{eff}}(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{\alpha}_\zeta} \quad \mathsf{CVaR}_\beta\left(\psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_\zeta)\right) + \mathsf{CVaR}_\beta\left(\phi\left(\boldsymbol{\alpha}_\zeta\right)\right)$$

Pessimistic estimate: take **dCVaR** of **free-energy** and **dissipation potential** + $\boldsymbol{\sigma}$ as first-stage variable

$$\begin{aligned} j^{\text{eff}}(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{\alpha}_\zeta, \boldsymbol{\varepsilon}_\zeta} \quad & \mathsf{dCVaR}_\beta\left(\psi(\boldsymbol{\varepsilon}_\zeta, \boldsymbol{\alpha}_\zeta)\right) + \mathsf{dCVaR}_\beta\left(\phi\left(\boldsymbol{\alpha}_\zeta\right)\right) \\ & \mathbb{E}\left[\boldsymbol{\varepsilon}_\zeta\right] = \boldsymbol{\varepsilon} \end{aligned}$$

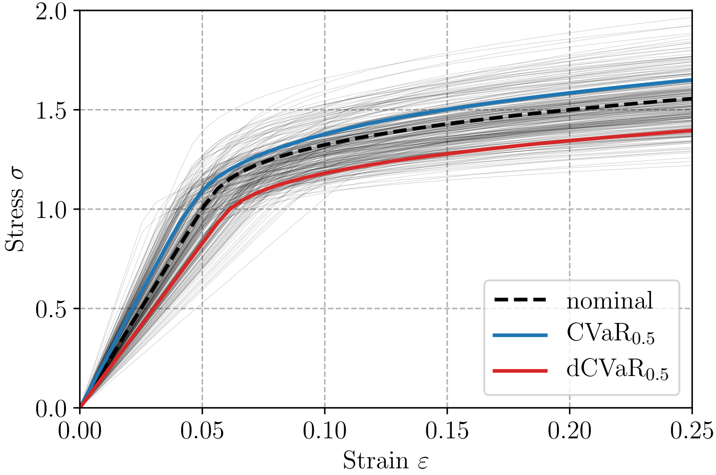
we essentially recover the previous primal/dual **risk-neutral** formulations when $\beta = 0$

Numerical results



risk-neutral case $\beta = 0$

Numerical results



moderate risk-aversion $\beta = 0.5$

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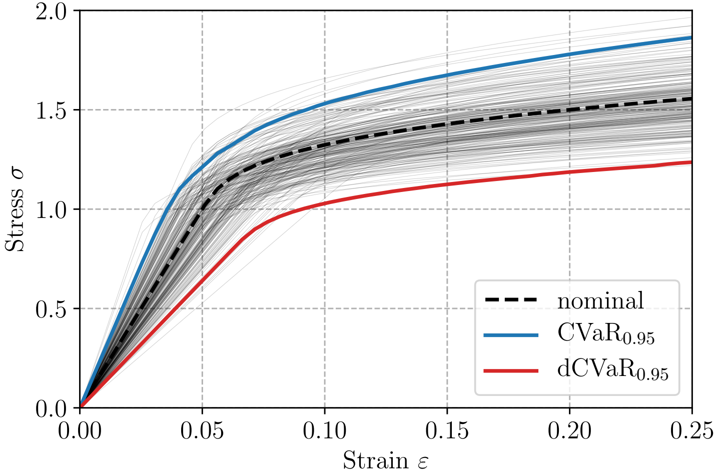
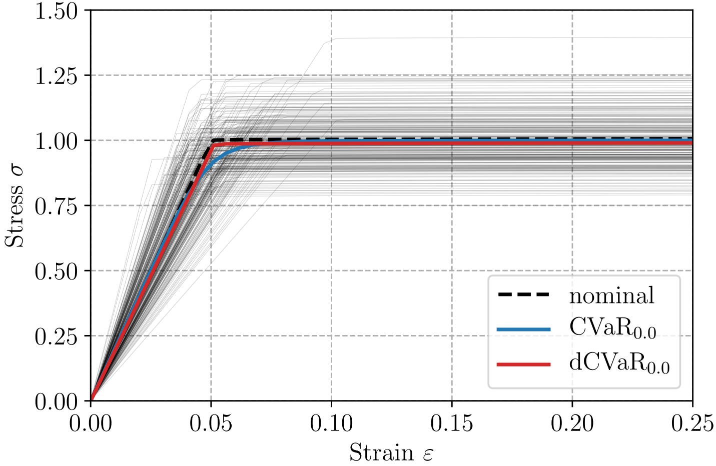


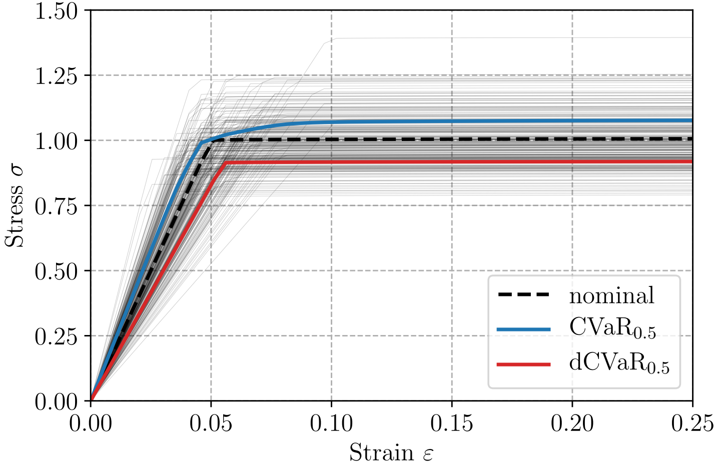
strong risk-aversion $\beta = 0.95$

Numerical results



risk-neutral case $\beta = 0$

Numerical results



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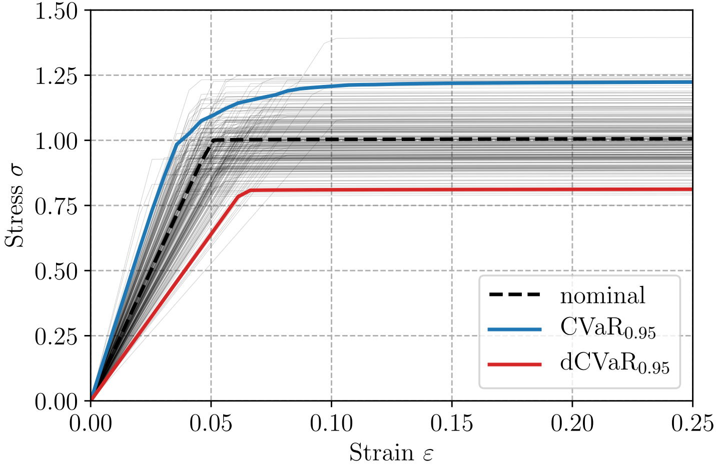

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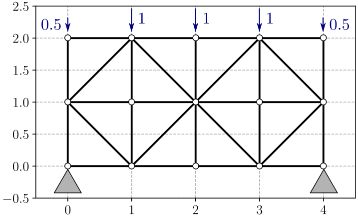
Outline

A truss example

Truss structure with members obeying stochastic elastoplastic hardening behaviour



Global nominal variational principle:

$$\boldsymbol{u}_{n+1}, \alpha_{n+1} = \underset{\boldsymbol{u}\in\mathcal{U}_{\text{ad}}, \boldsymbol{\alpha}}{\arg\inf} \int_\Omega \psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})\, d\Omega + \int_\Omega \phi(\boldsymbol{\alpha})\, d\Omega - \langle \boldsymbol{F}, \boldsymbol{u} \rangle$$

Risk-neutral case: effective global potential obtained from local effective properties

$$\mathcal{R}[J](\varepsilon) = \mathbb{E}\left[\int_\Omega j(\varepsilon;\zeta)\, d\Omega\right] = \int_\Omega \mathbb{E}[j](\varepsilon)\, d\Omega = \int_\Omega \mathcal{R}[j](\varepsilon)\, d\Omega$$

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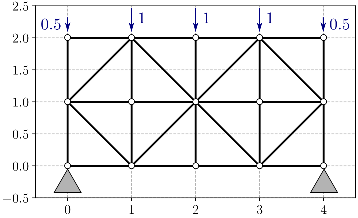

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Risk-averse case: **does not work**, $\mathcal{R} = \text{CVaR}/\text{dCVaR}$ not additive

Risk-averse stochastic programming formulation

Work directly on the **global** free-energy and dissipation potentials:

Optimistic formulation:

$$\boldsymbol{u}_{n+1}, \alpha_{\zeta,n+1} = \underset{\boldsymbol{u}\in\mathcal{U}_{\mathbf{ad}},\boldsymbol{\alpha}_\zeta}{\arg\inf} \;\mathrm{CVaR}_\beta\left(\Psi\right)\left(\varepsilon, \boldsymbol{\alpha}_\zeta\right) + \mathrm{CVaR}_\beta\left(\Phi\right)\left(\boldsymbol{\alpha}_\zeta\right) - \langle \boldsymbol{F}, \boldsymbol{u}\rangle$$

Pessimistic formulation:

$$\boldsymbol{u}_{n+1}, \alpha_{\zeta,n+1} = \underset{\boldsymbol{u}\in\mathcal{U}_{\mathbf{ad}},\boldsymbol{\alpha}_\zeta}{\arg\inf} \;\mathrm{dCVaR}_\beta\left(\Psi\right)\left(\varepsilon, \boldsymbol{\alpha}_\zeta\right) + \mathrm{dCVaR}_\beta\left(\Phi\right)\left(\boldsymbol{\alpha}_\zeta\right) - \langle \boldsymbol{F}, \boldsymbol{u}\rangle$$

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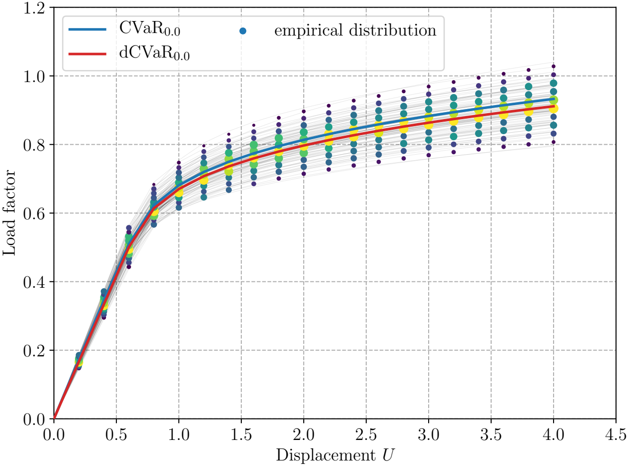
Both of them are **2-stage convex stochastic programming** problems which can be solved using Monte-Carlo sampling approximation

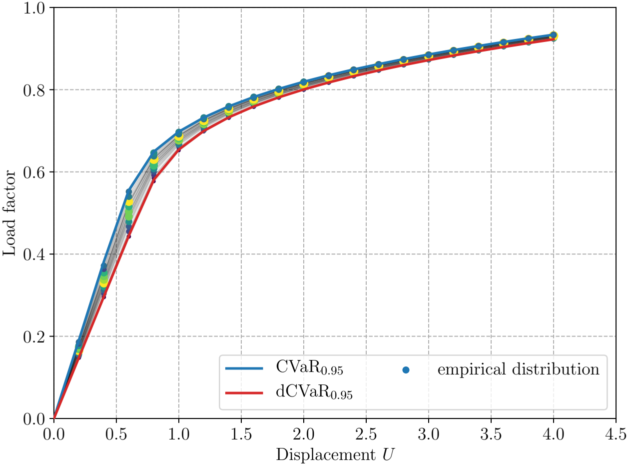
Illustrative application

Risk-neutral case



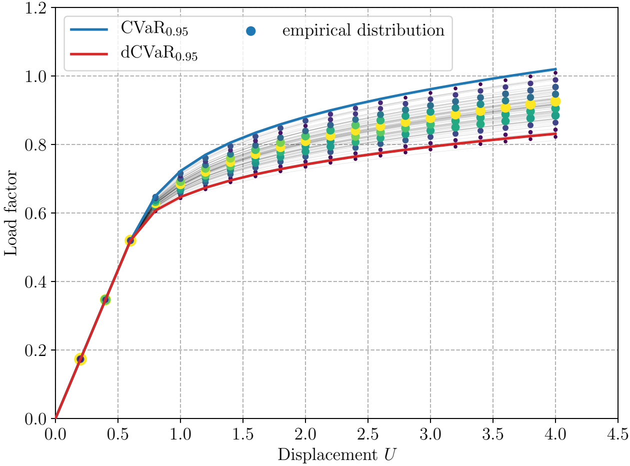
Illustrative application

Risk-averse case: uncertainty on **Young modulus only**



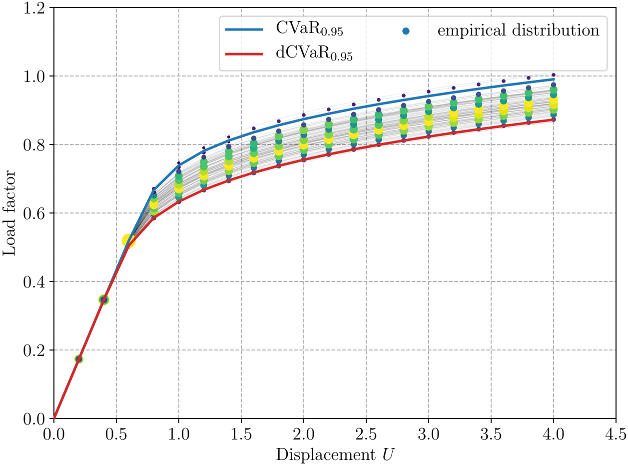
Illustrative application

Risk-averse case: uncertainty on **hardening modulus only**



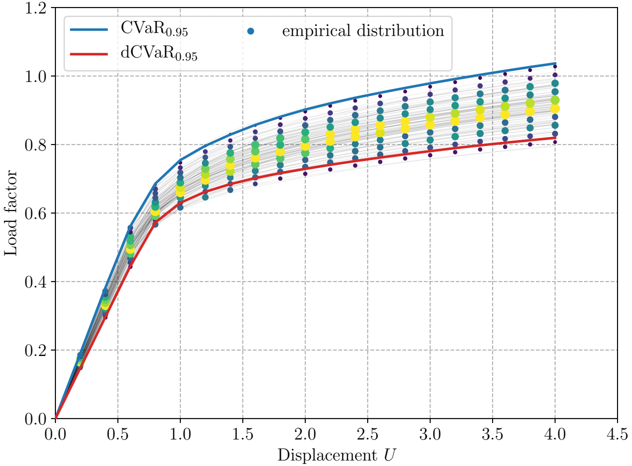
Illustrative application

Risk-averse case: uncertainty on **yield stress only**

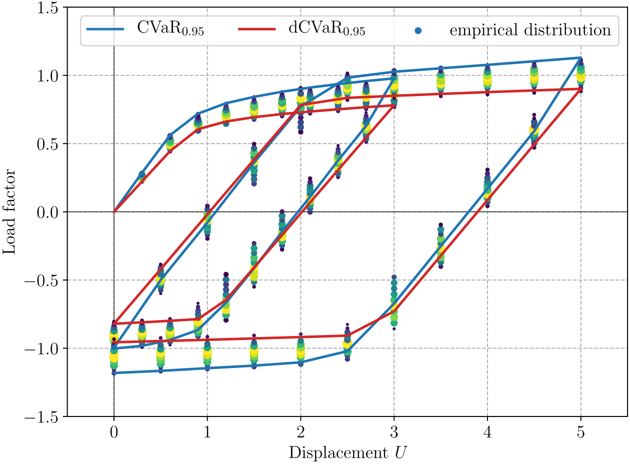


Illustrative application

Risk-averse case: combined uncertainty



Cyclic loading



Conclusions and Outlook

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- **consistent formulation** of **effective stochastic** behaviour in the GSM setting
- introduce the concepts of **risk measures** to characterize tail distribution behaviors
- dCVaR as a **novel coherent risk measure**
- extension of risk-neutral and risk-averse formulations to **global structural scale**

https://hal.science/hal-04076581

Outlook

- **reduce numerical cost** of Monte-Carlo sampling approximation
- **active scenarios strategies** in Newton-Raphson loop
- **non-linear decision rules**, **Polynomial Chaos expansions**, etc.

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Thank you for your attention !