A non-smooth tour in structural optimization

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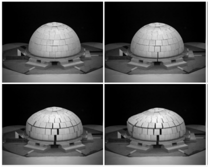
Fronts 2025

Variational problems

Nothing takes place in the world whose meaning is not that of some maximum or minimum. (Leonhard Euler).

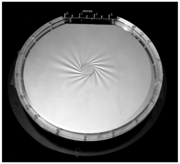
Collapse



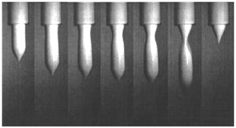
Masonry dome [Zessin, 2015]

Landslides [geologypage.com]

Thresholds

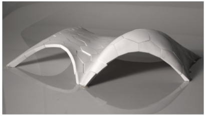


Membrane wrinkling [shellbuckling.com]



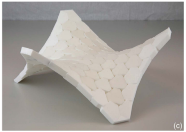
Mayonnaise drips [Coussot et al, 2005]

Optimization



Vault [P. Block group, ETH]

RC beam [S. Maitenaz]



Topology interlocking assembly [M. Pauly group, EPFL]

But also: finance, power networks, image processing, supply chain, machine learning, etc.

Outline

1. A primer on non-smooth optimization
2. Topology optimization of structural load-bearing capacity
3. Form-finding of funicular shells

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1. **A primer on non-smooth optimization**
2. Topology optimization of structural load-bearing capacity
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Convex variational problems

Differentiable case

$$\inf_{u\in V} J(u)$$

variational equality:
$D_u J(u, v) = 0 \quad \forall v \in V$

e.g. **potential energy**: $J(u) = \int_\Omega \psi(\nabla u)\,\mathrm{d}\Omega - \int_\Omega f u\,\mathrm{d}\Omega$

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Cone-constrained case

$$\begin{aligned} \inf_{u\in V} \quad & J(u) \\ \text{s.t.} \quad & u \in \mathcal{K} \end{aligned}$$

variational inequality:
$D_u J(u,v) \succeq_{\mathcal{K}^*} 0 \quad \forall v \in V$

Non-smooth case

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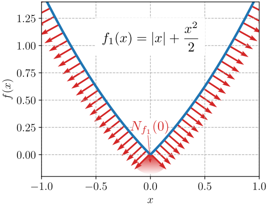
$D_u J(u,v) \ni 0 \quad \forall v \in V$



V.I. arise in presence of **inequality constraints** or **non-smooth** objective functions

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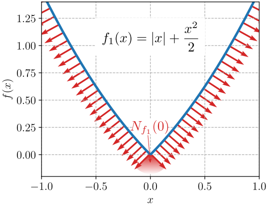


V.I. arise in presence of **inequality constraints** or **non-smooth** objective functions

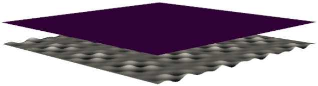
Typical feature: solution splits with

- points where constraints are **active**
- points where constraints are **inactive**

A simple concrete example: the obstacle problem

Elastic membrane deflecting against a **rigid obstacle**

$$\begin{aligned}\inf_{u\in V} \quad & \int_\Omega \frac{k}{2}\|\nabla u\|_2^2 \,\mathrm{d}\Omega - \int_\Omega fu \,\mathrm{d}\Omega \\ \text{s.t.} \quad & u \geq g \text{ in } \Omega\end{aligned}$$



Karush-Kuhn-Tucker optimality conditions:

$$\sup_{\lambda\geq 0}\inf_{u\in V} \int_\Omega \frac{k}{2}\|\nabla u\|_2^2 \,\mathrm{d}\Omega - \int_\Omega fu \,\mathrm{d}\Omega - \int_\Omega (u-g)\lambda \,\mathrm{d}\Omega$$

$$\Rightarrow \begin{cases} \displaystyle\int_\Omega k\nabla u \nabla v \,\mathrm{d}\Omega - \int_\Omega fv \,\mathrm{d}\Omega - \int_\Omega \lambda v \,\mathrm{d}\Omega = 0 \quad \forall v \in V & (u\text{-optimality}) \\ u - g \geq 0, \quad \lambda \geq 0, \quad (u-g)\lambda = 0 & (\text{complementarity}) \end{cases}$$

$u - g = 0 \Rightarrow$ **contact** v.s. $\lambda = 0 \Rightarrow$ **separation**

A more complex problem: Cheeger sets [Cheeger, 1969]

Find a subset ω with minimal ratio $\dfrac{\text{perimeter}}{\text{area}}$:

$$\inf_{\omega\subseteq\Omega} \frac{|\partial\omega|}{|\omega|} = \begin{array}{rl} \displaystyle\inf_{u\in V(\Omega)} & \displaystyle\int_\Omega \|\nabla u\|_2 \,\mathrm{d}\Omega \\ \text{s.t.} & \displaystyle\int_\Omega u\,\mathrm{d}\Omega = 1, \quad u = 0 \text{ on } \partial\Omega \end{array}$$

inherently non-smooth!

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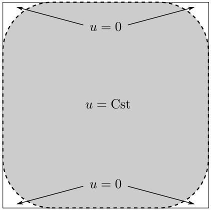

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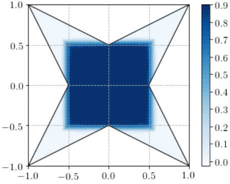
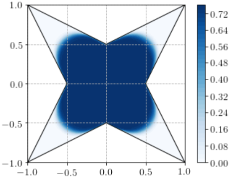
Dual problem:

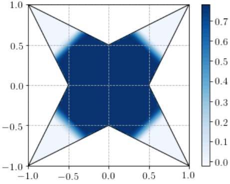
$$\begin{aligned} \sup_{\lambda,\boldsymbol{\sigma}} \quad & \lambda \\ \text{s.t.} \quad & \operatorname{div}\boldsymbol{\sigma} + \lambda = 0 \\ & \|\boldsymbol{\sigma}\| \leq 1 \end{aligned}$$



(a) $\|\nabla u\|_1$



(b) $\|\nabla u\|_2$



(c) $\|\nabla u\|_\infty$

Solvers for non-smooth problems

- **Bound-constrained problems**: relatively **easy** (projected gradient descent, bounded Newton, generic interior-point method)
- **Nonlinear complementarity problems**: e.g. contact, plasticity
  Find u, λ s.t. $f(u) \geq 0$, $\lambda \geq 0$ and $\lambda f(u) = 0$
  - regularization (**yuck!**)
  - active sets/semi-smooth Newton
  - use Fischer-Burmeister function with Newton method:

$$\Phi_{\mathsf{FB}}(x, y) = x + y - \sqrt{x^2 + y^2} = 0 \Longleftrightarrow x \geq 0, y \geq 0, xy = 0$$

- `PETSc TAO`: Toolkit for Advanced Optimization

PETSc TAO

- **Generic non-smooth problems**
  - **first-order** algorithms: Douglas-Rachford, ADMM, proximal methods
  - **second-order** algorithms: interior-point methods for linear/conic programming

Conic programming

Linear programming

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Conic programming

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$$\begin{aligned}\min_{\boldsymbol{x}} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in \mathcal{K}\end{aligned}$$

where $\mathcal{K}$ is a product of elementary cones e.g.

- positive orthants $\mathbb{R}^m_+$;
- Lorentz second-order cones: $\mathcal{Q}_m = \{\boldsymbol{z} = (z_0, \bar{\boldsymbol{z}}) \in \mathbb{R}_+ \times \mathbb{R}^{m-1} \text{ s.t. } \|\bar{\boldsymbol{z}}\|_2 \leq z_0\}$
- semi-definite cones $\mathcal{S}^+_m$, the cone of semi-definite positive $m \times m$ symmetric matrices;
- power cones, exponential cones, etc.

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The magic cone family [Juditsky & Nemirovski, 2021]

very large modelling power of **convex** functions and constraints

$$\begin{aligned} f(\boldsymbol{x}) = \min_{\boldsymbol{y}} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} + \boldsymbol{d}^\mathsf{T}\boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{b}_l \leq \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{b}_u \\ & \boldsymbol{y} \in \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_p \end{aligned} \qquad \text{(conic representation)}$$

Solvers

primal-dual interior point methods: very efficient and robust (20-30 iterations)

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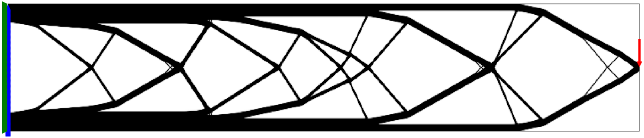
3. Form-finding of funicular shells

Topology optimization : elastic setting

Find $\Omega \subseteq \mathcal{D}$ minimizing the **elastic compliance** at fixed volume [Allaire, 2002]:



[TopOpt in Python, DTU]

$$\begin{aligned} \min_{\Omega,\boldsymbol{u}} \quad & \int_{\partial\Omega} \boldsymbol{T}\cdot\boldsymbol{u}\,\mathrm{dS} \\ \text{s.t.} \quad & \boldsymbol{\sigma} = \mathbb{C} : \nabla\boldsymbol{u} \text{ in } \Omega \\ & \operatorname{div}\boldsymbol{\sigma} = 0 \quad \text{in } \Omega \\ & \boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{T} \quad \text{on } \partial\Omega_N \\ & |\Omega| \leq \eta|\mathcal{D}| \end{aligned}$$

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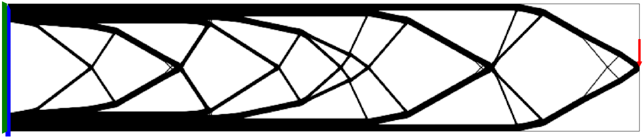

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Density-based formulation

$$
\begin{aligned}
\min_{\rho, \boldsymbol{u}} \quad & \int_{\partial \mathcal{D}} \boldsymbol{T} \cdot \boldsymbol{u} \, \mathrm{dS} \\
\text{s.t.} \quad & \boldsymbol{\sigma} = \mathbb{C}(\rho) : \nabla \boldsymbol{u} \text{ in } \mathcal{D} \\
& \operatorname{div} \boldsymbol{\sigma} = 0 \quad \text{in } \mathcal{D} \\
& \boldsymbol{\sigma} \boldsymbol{n} = \boldsymbol{T} \quad \text{on } \partial \mathcal{D}_N \\
& \int_{\mathcal{D}} \rho \, \mathrm{d}\Omega \leq \eta |\mathcal{D}| \\
& 0 \leq \rho(\boldsymbol{x}) \leq 1
\end{aligned}
$$

$\Rightarrow$ **non-convex** problem, iterative procedure

e.g. SIMP method [Bendsoe and Kikuchi, 1988] : $\mathbb{C}(\rho) = \rho^p \mathbb{C}_0$ with $p > 1$

Topology optimization : elastic setting

Limitations of compliance minimization with an **elastic behaviour**

- in practice, materials are non-linear (plasticity, tension/compression, etc.)
- does not account for stress limits
- not relevant for optimizing reinforcements e.g. steel in reinforced concrete

Extensions to **nonlinear behaviours**:

- local stress constraints make TopOpt formulations more difficult (non smoothness) [Duysinx and Bendoe, 1998]
- sensitivities (or shape derivatives) are often computed using regularized behaviours [Maury et al., 2018]
- need to optimize with respect to the whole loading path

Here, we want to optimize with the structure **limit load** rather than elastic compliance using **limit analysis theory**

Limit analysis theory: a convex optimization formulation

Collapse = there exist no **internal stress field** satisfying both **equilibrium** and **strength** conditions [Hill, 1950; Salençon, 1983]

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- **Stress field**: a symmetric 2nd-rank tensor $\boldsymbol{\sigma}(\boldsymbol{x})$ for 2D/3D solids
- **Equilibrium** with respect to a given loading :

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} &= 0 \quad \text{for } \boldsymbol{x} \in \Omega \\ \boldsymbol{\sigma} \boldsymbol{n} &= \lambda \boldsymbol{T} \quad \text{for } \boldsymbol{x} \in \partial\Omega_N \end{aligned}$$

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- **Strength condition**: G **convex** set containing 0

$$\begin{aligned} & \boldsymbol{\sigma}(\boldsymbol{x}) \in G(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega \\ \Leftrightarrow \quad & g_G(\boldsymbol{\sigma}) \leq 1 \end{aligned}$$

g_G Minkowski functional (gauge) of G

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Collapse load

Find the **maximum** load multiplier λ such that such a stress field exists

Convex optimization formulation

Continuous problem:

$$\begin{aligned}\lambda^+ = \max_{\lambda, \boldsymbol{\sigma} \in \mathcal{W}} \quad & \lambda \\ \text{s.t.} \quad & \operatorname{div} \boldsymbol{\sigma} = 0 && \forall \boldsymbol{x} \in \Omega \\ & \boldsymbol{\sigma} \boldsymbol{n} = \lambda \boldsymbol{T} && \forall \boldsymbol{x} \in \partial\Omega_N \\ & g_G(\boldsymbol{\sigma}) \leq 1 \end{aligned}$$

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Discrete (e.g. finite-element) formulation:

$$
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\text{s.t.} \quad & \boldsymbol{H}\boldsymbol{\sigma} + \lambda \boldsymbol{F} = 0 \\
& g_G(\boldsymbol{\sigma}_k) \leq 1 && \forall k = 1, \dots, N
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convex optimization problems

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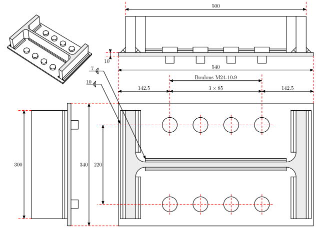
convex optimization problems

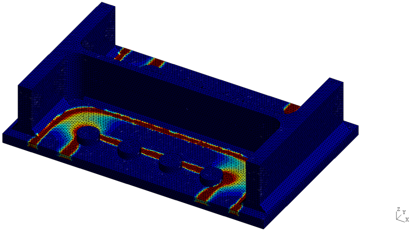
usually G (thus also g_G) has a simple geometrical shape: ellipsoid, cone, etc.

$\Rightarrow$ **conic programming solvers** e.g. `MOSEK`

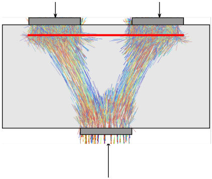
Examples

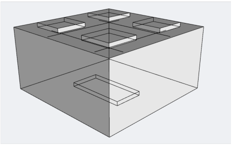
Bolted column base plate [C. El Boustani thesis]:

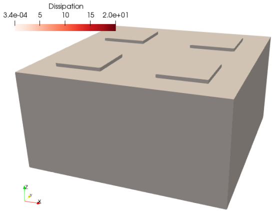




Reinforced concrete bridge pier cap [H. Vincent thesis]:







Maximizing the limit load

Objective : Find $\Omega \subseteq \mathcal{D}$ with **maximum limit load** for a given volume level η:

Proposed formulation:

$$\begin{aligned} \lambda^+(\eta) = \max_{\lambda,\boldsymbol{\sigma},\Omega} \quad & \lambda \\ \text{s.t.} \quad & \operatorname{div}\boldsymbol{\sigma} = 0 && \text{in } \Omega \\ & \boldsymbol{\sigma}\boldsymbol{n} = \lambda \boldsymbol{T} && \text{in } \partial\Omega_N \\ & g_G(\boldsymbol{\sigma}) \leq 1 && \text{in } \Omega \\ & |\Omega| \leq \eta|\mathcal{D}| \end{aligned}$$

Maximizing the limit load

Objective : Find $\Omega \subseteq \mathcal{D}$ with **maximum limit load** for a given volume level η:

extension by $\boldsymbol{\sigma} = 0$ outside Ω

$$
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\lambda^+(\eta) = \displaystyle\max_{\lambda,\boldsymbol{\sigma},\rho} & \lambda & \\
\text{s.t.} & \operatorname{div}\boldsymbol{\sigma} = 0 & \text{in } \mathcal{D} \\
& \boldsymbol{\sigma}\boldsymbol{n} = \lambda\boldsymbol{T} & \text{in } \partial\mathcal{D}_N \\
& g_G(\boldsymbol{\sigma}) \leq \rho & \text{in } \mathcal{D} \\
& \int_{\mathcal{D}} \rho \, \mathrm{d}\Omega \leq \eta|\mathcal{D}| & \\
& \rho \in \{0; 1\} &
\end{array}
$$

ρ being the characteristic function of Ω

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problem **convexification** `(LOAD-MAX)`

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& g_G(\boldsymbol{\sigma}) \leq \rho && \text{in } \mathcal{D} \\
& \textstyle\int_{\mathcal{D}} \rho \, d\Omega \leq \eta|\mathcal{D}| \\
& \rho \in [0; 1]
\end{aligned}
$$

Properties:

- $g_G(\boldsymbol{\sigma}) \leq \rho$ is a **convex constraint** in $(\boldsymbol{\sigma}, \rho)$
- akin to a limit analysis problem with an additional scalar variable ρ
- $\lambda^+(\eta = 1)$: limit load associated with $\mathcal{D}$

Numerical implementation

Penalisation procedure

- **continuation** on $g_G(\boldsymbol{\sigma}) \leq \rho^p$ with $p \to p_{\text{max}} > 1$ (SIMP-like)
  $\Rightarrow \rho$ is driven towards 0 or 1 (**truss**-like topology)
  At step n, linearisation around ρ_{n-1}:

$$\rho^{p_n} \approx \rho_{n-1}^{p_n} + p_n \rho_{n-1}^{p_n-1}(\rho - \rho_{n-1}) = a_n + b_n \rho$$

  typically $p_{\text{max}} = 3$, 20 iterations

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  At step n, linearisation around ρ_{n-1}:

$$\rho^{p_n} \approx \rho_{n-1}^{p_n} + p_n \rho_{n-1}^{p_{n-1}}(\rho - \rho_{n-1}) = a_n + b_n \rho$$

  typically $p_{\max} = 3$, 20 iterations
- **localisation limiter** by gradient-control constraint

$$\|\nabla \rho\|_2 \leq 1/\ell$$

  with ℓ a regularization length (a few times larger than h) $\Rightarrow$ **mesh-independency**

Numerical implementation

Penalisation procedure

- **continuation** on $g_G(\boldsymbol{\sigma}) \leq \rho^p$ with $p \to p_{\max} > 1$ (SIMP-like)
  $\Rightarrow \rho$ is driven towards 0 or 1 (**truss**-like topology)
  At step n, linearisation around ρ_{n-1}:

$$\rho^{p_n} \approx \rho_{n-1}^{p_n} + p_n \rho_{n-1}^{p_{n-1}}(\rho - \rho_{n-1}) = a_n + b_n \rho$$

  typically $p_{\max} = 3$, 20 iterations
- **localisation limiter** by gradient-control constraint

$$\|\nabla \rho\|_2 \leq 1/\ell$$

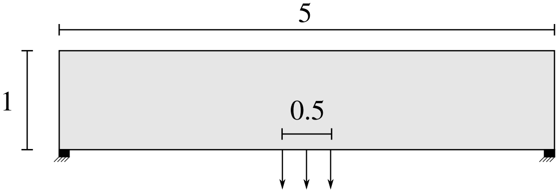
  with ℓ a regularization length (a few times larger than h) $\Rightarrow$ **mesh-independency**

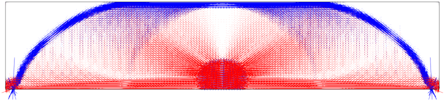
(a) $\eta = 0.2$, $\lambda^+ = 0.36\Lambda^+$

(b) $\eta = 0.3$, $\lambda^+ = 0.55\Lambda^+$

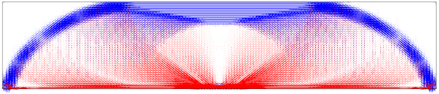
(c) $\eta = 0.4$, $\lambda^+ = 0.77\Lambda^+$

Material with asymmetric strengths





(a) $f_c/f_t = 10$

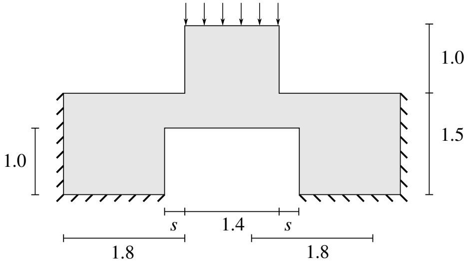


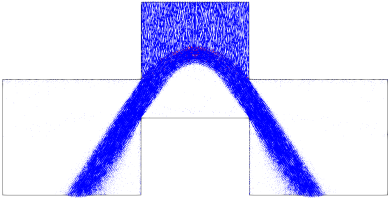
(b) $f_c/f_t = 0.1$

Principal stresses (compression/traction)

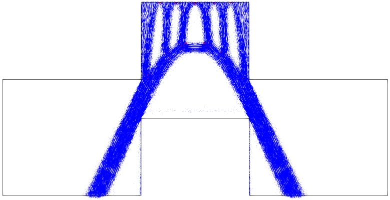
Material without tensile strength

ex: rock or masonry structures





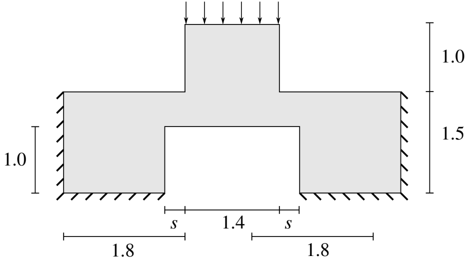
(a) Before penalization



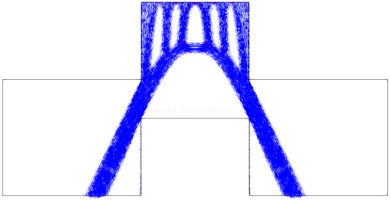
(b) After penalization

Material without tensile strength

ex: rock or masonry structures



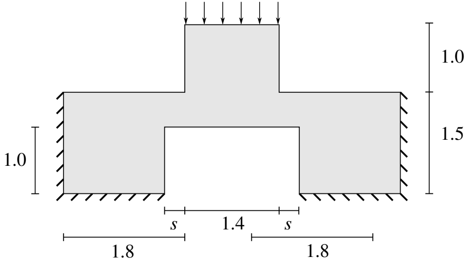
(a) The Passion Façade

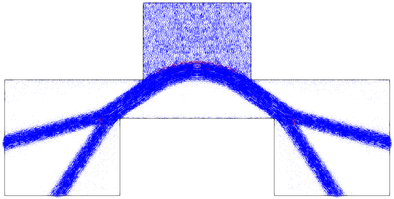


(b) After penalization

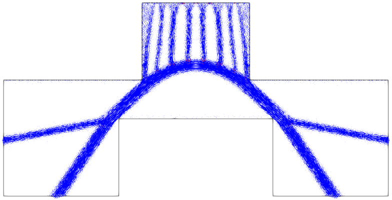
Material without tensile strength

ex: rock or masonry structures



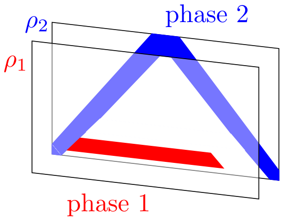


(a) Before penalization



(b) After penalization

Topology optimization with two phases

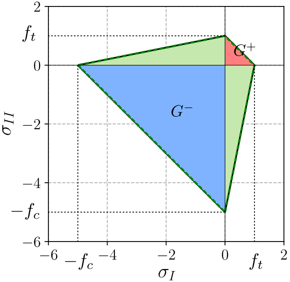


we want to optimize independently **two phases** (+ void)
e.g. **steel** and **concrete**, **tension** and **compression**

Strength condition

$$\boldsymbol{\sigma} \in G(\rho_1, \rho_2) \Leftrightarrow \exists \boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2 \text{ s.t. } \begin{cases} \boldsymbol{\sigma} = \boldsymbol{\sigma}^1 + \boldsymbol{\sigma}^2 \\ \boldsymbol{\sigma}^1 \in \rho_1 G^1 \\ \boldsymbol{\sigma}^2 \in \rho_2 G^2 \end{cases}$$

with $\rho_1 + \rho_2 \leq 1$

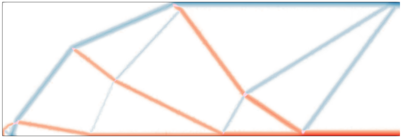


Problem formulation

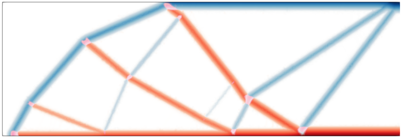
$$
\begin{aligned}
\lambda^+ = \quad & \max_{\lambda,\boldsymbol{\sigma^1},\boldsymbol{\sigma^2},\rho_1,\rho_2} \lambda \\
\text{s.t.} \quad & \operatorname{div}(\boldsymbol{\sigma}^1 + \boldsymbol{\sigma}^2) = 0 \\
& (\boldsymbol{\sigma}^1 + \boldsymbol{\sigma}^2)\boldsymbol{n} = \lambda \boldsymbol{T} \\
& \boldsymbol{\sigma}^1 \in \rho_1 G^1 \\
& \boldsymbol{\sigma}^2 \in \rho_2 G^2 \\
& \int_{\mathcal{D}} (\rho_1 + \rho_2)\, d\Omega \leq \eta |\mathcal{D}| \\
& 0 \leq \rho_1 \leq 1 \\
& 0 \leq \rho_2 \leq 1 \\
& \rho_1 + \rho_2 \leq 1
\end{aligned}
$$

MBB beam

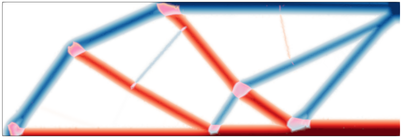
L_1-Rankine with **pure compression** G^- and **pure tension** G^+



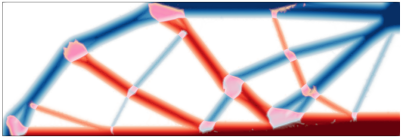
(a) $\eta = 0.05$



(b) $\eta = 0.1$

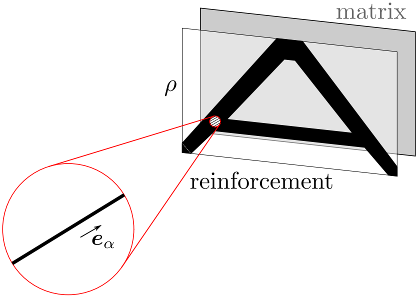


(c) $\eta = 0.2$



(d) $\eta = 0.3$

Anisotropic strength condition for reinforcements

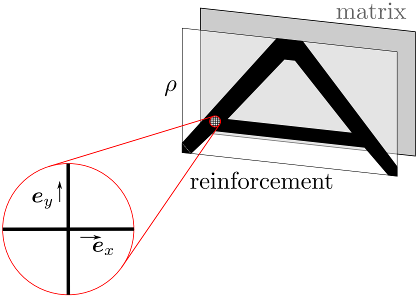


Uniaxial reinforcements

only one **known** direction $\boldsymbol{e}_\alpha$

$$G^{\mathsf{r}} = \{\sigma^{\mathsf{r}}\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\alpha \text{ s.t. } -f_c^{\mathsf{r}} \leq \sigma^{\mathsf{r}} \leq f_t^{\mathsf{r}}\}$$

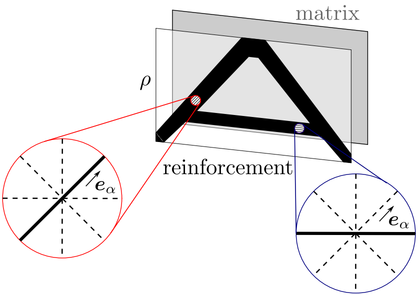
Anisotropic strength condition for reinforcements



Reinforcements along x and y

$$G^{\mathsf{r}} = \{\sigma^{\mathsf{r},x} \boldsymbol{e}_x \otimes \boldsymbol{e}_x + \sigma^{\mathsf{r},y} \boldsymbol{e}_y \otimes \boldsymbol{e}_y$$
$$\text{s.t.} \quad -f_c^{\mathsf{r}} \leq \sigma^{\mathsf{r},x}, \sigma^{\mathsf{r},y} \leq f_t^{\mathsf{r}}\}$$

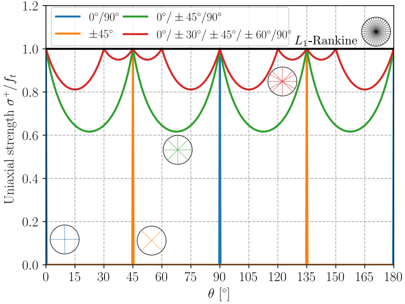
Anisotropic strength condition for reinforcements



Distributed uniaxial reinforcements

$$\boldsymbol{\sigma}^{\mathsf{r}} \in G^{\mathsf{r}} \Leftrightarrow \exists \sigma^{\mathsf{r},\alpha}, \zeta_\alpha \text{ s.t. } \begin{cases} \boldsymbol{\sigma}^{\mathsf{r}} = \sum_{\alpha\in\mathcal{A}} \zeta_\alpha \sigma^{\mathsf{r},\alpha} \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\alpha & \\ -f_c^{\mathsf{r}} \leq \sigma^{\mathsf{r},\alpha} \leq f_t^{\mathsf{r}} & \forall \alpha \in \mathcal{A} \\ \sum_{\alpha\in\mathcal{A}} \zeta_\alpha = 1, \quad \zeta_\alpha \in \{0; 1\} & \forall \alpha \in \mathcal{A} \end{cases}$$

Anisotropic strength condition for reinforcements

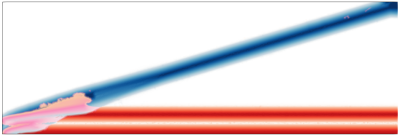


Distributed uniaxial reinforcements (convexification)

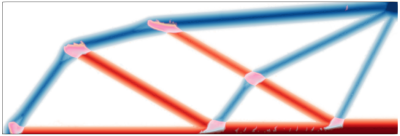
$$\boldsymbol{\sigma}^{\mathrm{r}} \in G^{\mathrm{r}} \Leftrightarrow \exists \sigma^{\mathrm{r},\alpha}, \zeta_\alpha \text{ s.t. } \begin{cases} \boldsymbol{\sigma}^{\mathrm{r}} = \displaystyle\sum_{\alpha \in \mathcal{A}} \zeta_\alpha \sigma^{\mathrm{r},\alpha} \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\alpha & \\ -f_c^{\mathrm{r}} \leq \sigma^{\mathrm{r},\alpha} \leq f_t^{\mathrm{r}} & \forall \alpha \in \mathcal{A} \\ \displaystyle\sum_{\alpha \in \mathcal{A}} \zeta_\alpha = 1, \quad \zeta_\alpha \in [0;1] & \forall \alpha \in \mathcal{A} \end{cases}$$

MBB beam with specific orientations

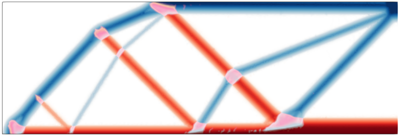
specific orientations for one phase e.g. the traction phase



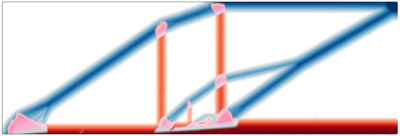
(a) Orientations along 0°



(b) Orientations along 0° and ±30°



(c) Orientations along 0° and ±45°



(d) Orientations along 0° and 90°

A deep-beam structure

The proposed methodology is linked with the use of **strut-and-tie models** for the design of reinforced-concrete structures

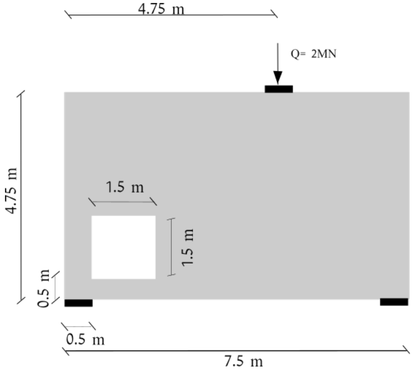


Figure: Deep beam with 25 cm width, analyzed in [Muttoni et al., 2015]

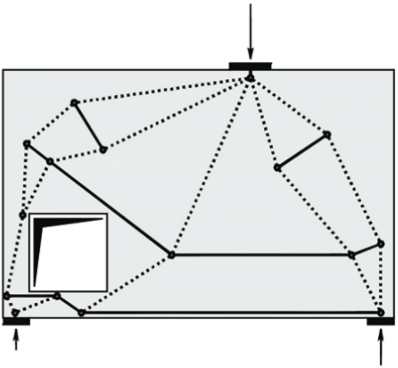


Figure: Strut-and-tie model proposed in [Muttoni et al., 2015]

A deep-beam structure

The proposed methodology is linked with the use of **strut-and-tie models** for the design of reinforced-concrete structures

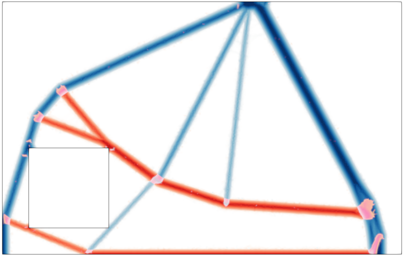


Figure: Isotropic reinforcements

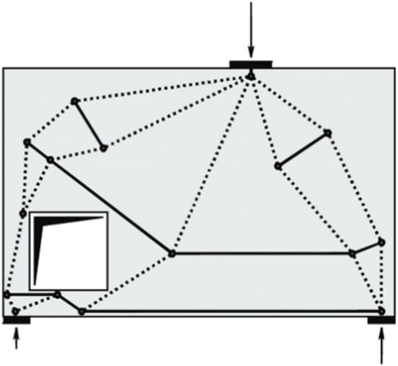


Figure: Strut-and-tie model proposed in [Muttoni et al., 2015]

A deep-beam structure

The proposed methodology is linked with the use of **strut-and-tie models** for the design of reinforced-concrete structures

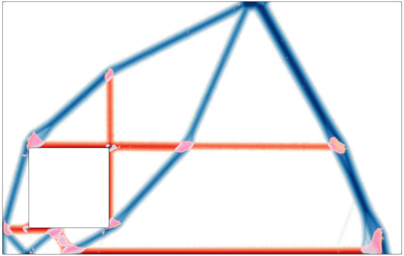


Figure: Orthogonal reinforcements

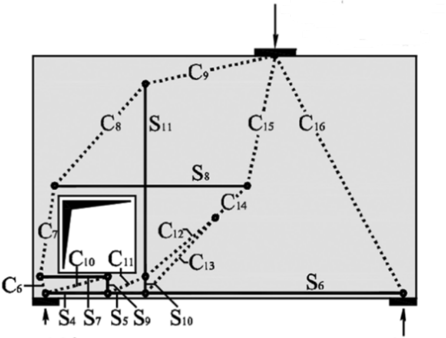
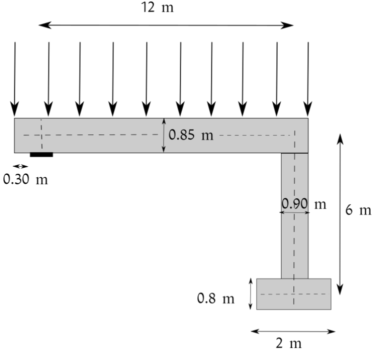
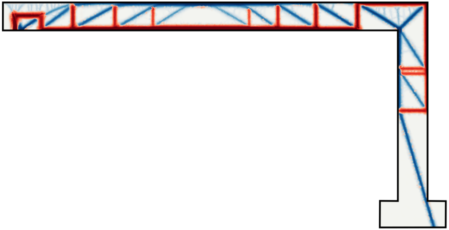


Figure: Strut-and-tie model with orthogonal reinforcements proposed in [Muttoni et al., 2015]

Beam and column





Outline

1 A primer on non-smooth optimization

2 Topology optimization of structural load-bearing capacity

3 Form-finding of funicular shells

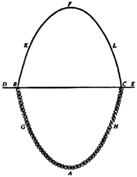
Form finding in structural mechanics

Hanging-chain inversion

Ut pendet continuum flexile, sic stabit contiguum rigidum inversum.

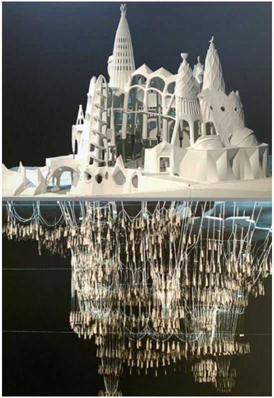
As hangs the flexible line, so but inverted will stand the rigid arch.

[Robert Hooke]



[Poleni, 1748]

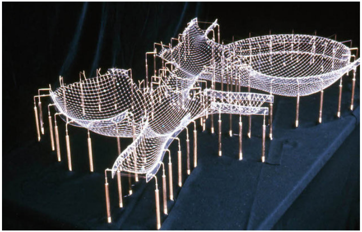
Antoni Gaudí



[http://www.artstudio.org]

Form finding in structural mechanics

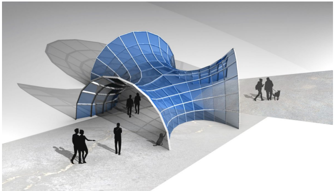
Frei Otto [Ian Liddell, Frei Otto and the development of gridshells, 2015]



Hanging chain scale model

Mannheim Multihalle Gridshell

Soap films : constant mean curvature, funicular under normal pressure



[X. Tellier, laboratoire Navier]

Membranes in structural engineering

Tension membrane roofing structures

[ThinkShell, ENPC]

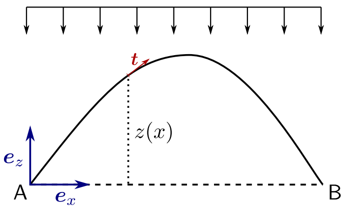
Olympiapark, Munich

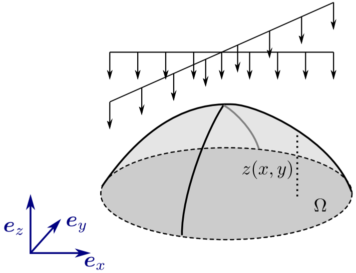
Inflatable structures/formwork

[fstructures.com]

[C. Boutemy, laboratoire Navier]

Form-finding with Geometric Stiffness Optimization





Reference (flat) domain Ω modeled as a **fictitious membrane**

- no elasticity, only initial **prestress** $\boldsymbol{\Sigma}(\boldsymbol{x})$
- free energy depends only on prestress (geometric stiffness):

$$\Psi(\boldsymbol{E}) = \boldsymbol{\Sigma} : \boldsymbol{E}$$

  where $\boldsymbol{E} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}} + \nabla \boldsymbol{u}^{\mathsf{T}} \nabla \boldsymbol{u})$ is the Green-Lagrange strain
- **key observation**: for positive prestress $\boldsymbol{\Sigma} \succeq 0$

$$\Psi(\nabla \boldsymbol{u}) \text{ is } \textbf{convex}$$

Geometric Stiffness Optimization formulation

membrane geometric stiffness:

$$\mathcal{S}(\boldsymbol{u},\boldsymbol{\Sigma}) = \int_\Omega \frac{1}{2}\boldsymbol{\Sigma} : \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\mathsf{T} + \nabla \boldsymbol{u}^\mathsf{T}\nabla \boldsymbol{u}\right) \,\mathrm{d}\Omega$$

membrane geometric compliance:

$$\mathcal{C}(\boldsymbol{\Sigma}) = \sup_{\boldsymbol{u}\in\mathcal{V}} \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}\Omega - \mathcal{S}(\boldsymbol{u},\boldsymbol{\Sigma})$$

Geometric Stiffness Optimization formulation

membrane geometric stiffness:

$$\mathcal{S}(\boldsymbol{u}, \boldsymbol{\Sigma}) = \int_\Omega \frac{1}{2}\boldsymbol{\Sigma} : \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathsf{T}} + \nabla \boldsymbol{u}^{\mathsf{T}} \nabla \boldsymbol{u}\right) \,\mathrm{d}\Omega$$

membrane geometric compliance:

$$\mathcal{C}(\boldsymbol{\Sigma}) = \sup_{\boldsymbol{u}\in\mathcal{V}} \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}\Omega - \mathcal{S}(\boldsymbol{u}, \boldsymbol{\Sigma})$$

Using **convex duality**, with $\boldsymbol{P}$ the 1^{st} Piola-Kirchhoff stress:

$$\begin{aligned} \mathcal{C}(\boldsymbol{\Sigma}) = \inf_{\boldsymbol{P}} \quad & \int_\Omega \frac{1}{2}(\boldsymbol{P} - \boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1} : (\boldsymbol{P} - \boldsymbol{\Sigma})\,\mathrm{d}\Omega \\ \text{s.t.} \quad & \operatorname{div} \boldsymbol{P} + \boldsymbol{f} = 0 \quad \text{on } \Omega \\ & \boldsymbol{P}\boldsymbol{n} = 0 \quad \text{on } \partial\Omega_{\mathsf{N}} \end{aligned}$$

Geometric Stiffness Optimization formulation

membrane geometric stiffness:

$$\mathcal{S}(\boldsymbol{u}, \boldsymbol{\Sigma}) = \int_\Omega \frac{1}{2}\boldsymbol{\Sigma} : \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\mathsf{T} + \nabla \boldsymbol{u}^\mathsf{T} \nabla \boldsymbol{u}\right) \,\mathrm{d}\Omega$$

membrane geometric compliance:

$$\mathcal{C}(\boldsymbol{\Sigma}) = \sup_{\boldsymbol{u}\in\mathcal{V}} \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}\Omega - \mathcal{S}(\boldsymbol{u}, \boldsymbol{\Sigma})$$

Using **convex duality**, with $\boldsymbol{P}$ the 1st Piola-Kirchhoff stress:

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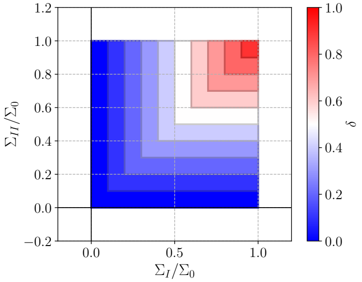
we now look for the **optimal prestress** $\boldsymbol{\Sigma}$ which **minimizes** the geometric compliance $\mathcal{C}(\boldsymbol{\Sigma})$

$$\begin{aligned} \inf_{\boldsymbol{\Sigma}\in S_{\mathbf{ad}}} \mathcal{C}(\boldsymbol{\Sigma}) = \inf_{\boldsymbol{\Sigma}, \boldsymbol{P}} \quad & \int_\Omega \frac{1}{2}(\boldsymbol{P} - \boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1} : (\boldsymbol{P} - \boldsymbol{\Sigma}) \,\mathrm{d}\Omega \\ \text{s.t.} \quad & \operatorname{div} \boldsymbol{P} + \boldsymbol{f} = 0 \quad \text{on } \Omega \\ & \boldsymbol{P}\boldsymbol{n} = 0 \quad \text{on } \partial\Omega_\mathsf{N} \\ & \boldsymbol{\Sigma} \succeq 0 \\ & g(\boldsymbol{\Sigma}) \leq \Sigma_0 \end{aligned}$$

Prestress cost functions

Without any constraint on the prestress $\boldsymbol{\Sigma}$, optimal structures would have infinite prestress $\Rightarrow$ prestresses are constrained to belong to a convex set $g(\boldsymbol{\Sigma}) \leq \Sigma_0$ e.g.

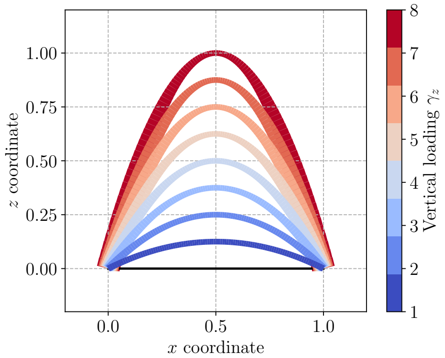
- uniform isotropic prestress $\boldsymbol{\Sigma} = \Sigma_0 \mathbf{1}$
- bounded prestress $\|\boldsymbol{\Sigma}\|_\infty = \max\{\Sigma_I, \Sigma_{II}\} \leq \Sigma_0$
- L_1-norm bounded prestress $\|\boldsymbol{\Sigma}\|_1 = \Sigma_I + \Sigma_{II} \leq \Sigma_0$
- bounded prestress around an isotropic state



δ varies the diameter of the admissible set

Optimal 1D arches

Optimal arches for constant prestress with $\Sigma_0 = 1$



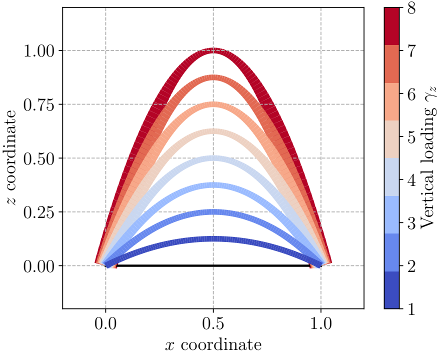
Vertical loading only ($\gamma_x = 0$)

Saint-Louis (Missouri) Gateway Arch

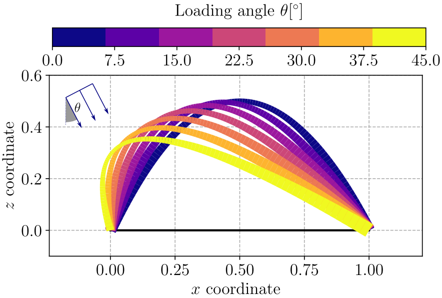
The thickness indicates the distribution of the cross-section $A(x)$.

Optimal 1D arches

Optimal arches for constant prestress with $\Sigma_0 = 1$



Vertical loading only ($\gamma_x = 0$)

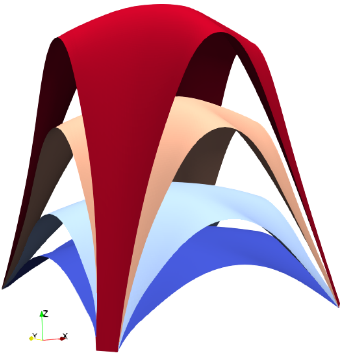


Inclined loading with $\gamma_x = \gamma \sin\theta$ and $\gamma_z = -\gamma\cos\theta$

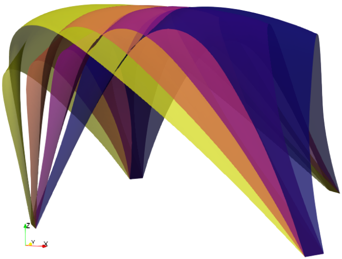
The thickness indicates the distribution of the cross-section $A(x)$.

Square vaults

Optimal square vaults ($\delta = 1$)

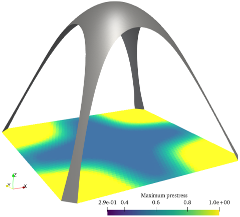


Vertical loading only $\gamma_z = 0.5, 1, 2, 3$ (from blue to red)

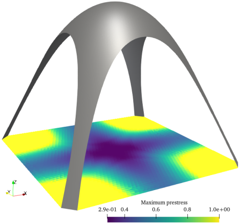


Vertical and horizontal loading $\gamma_z = 2$ and $\gamma_x = 0, 0.5, 1, 1.5$ (from blue to yellow)

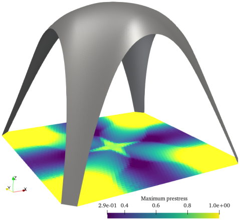
Square vaults



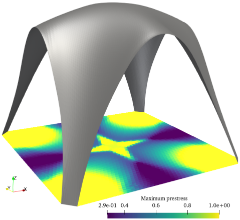
(a) $\delta = 0.5$



(b) $\delta = 0.7$



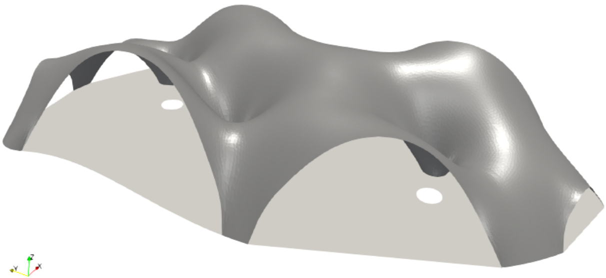
(c) $\delta = 0.9$

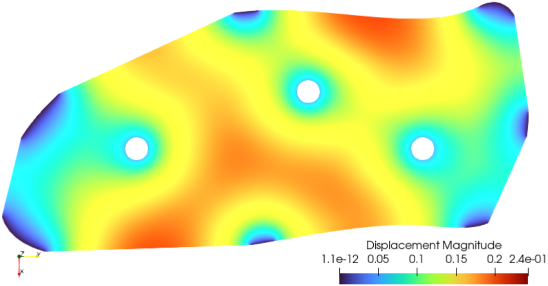


(d) $\delta = 1$

Optimal square vaults when varying δ. Maximum prestress Σ_I/Σ_0 is represented below the surface on the reference domain.

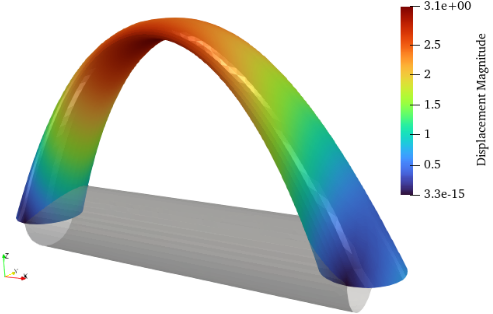
Complex structure



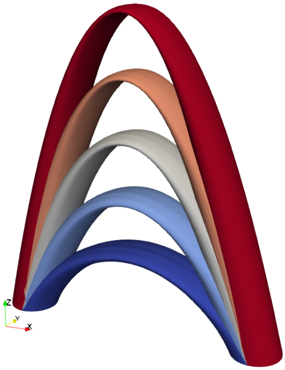


Non-planar reference domain

Ω is now a cylinder (in gray)
As bcs, we impose a rigid body rotation of both ends of angle $\pm\pi/2$, enforcing circular cross-sections to lie within the $z = 0$ plane



Reference cylinder and optimal shape for $\gamma_z = 0.5$



Optimal shapes for γ_z varying from 0.1 to 2 (blue to red)

Resources

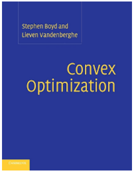
Modeling languages `cvxpy` (Python), `JuMP` (Julia)

```
import cvxpy as cp
import numpy as np

# generate random data
m, n = 200, 10
C = np.random.rand(m, n)
d = np.random.rand(m)

# define problem
x = cp.Variable(n)
objective = 0.5*cp.norm2(C*x-d)**2
constraints = [cp.norm1(x) <= 1]
problem = cp.Problem(cp.Minimize(objective), constraints)
problem.solve(solver=cp.MOSEK)
```

Books and **lectures** [Boyd & Vandenberghe, Convex Optimization]



Stanford University Convex Optimization Group
https://github.com/cvxgrp

PDE-based optimization: dolfinx_optim package

FEniCS PROJECT

$$\min_{u \in V} \int_\Omega F(u)\, d\Omega$$

FE discretization $\Longrightarrow$ canonicalization

MOSEK

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathcal{K} \end{aligned}$$