Variational principles in nonlinear mechanics using convex optimization and automated numerical tools

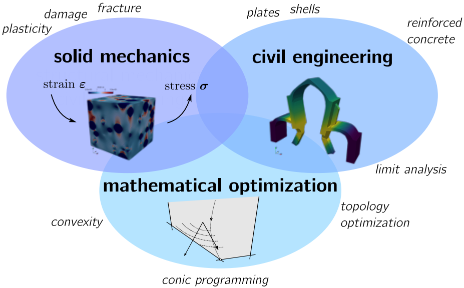
Jeremy Bleyer

Habilitation à Diriger des Recherches
November, 22nd 2024

About me

Curriculum

Research interests



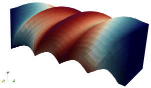
Teaching

$\approx$ 120 hours/year

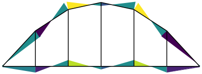
École Polytechnique

Structural mechanics

Solid mechanics

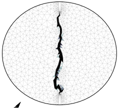


FEM for civil engineering



python

Damage mechanics



École des Ponts

FEniCS project

Teaching

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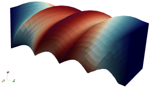

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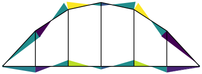

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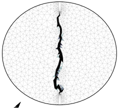
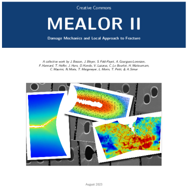
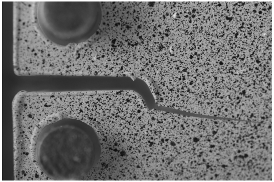
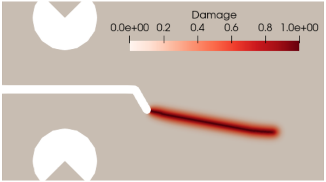


École des Ponts

MEALOR II Summer school: *physics and mechanics of brittle and ductile fracture*
collective open-access book







PhD students

Academic/Industrial

			defended in
H. Vincent STRAINS	K. Cascavita with A. Ern LABEX MMCD	C. El Boustani STRAINS	2018 – 2020
L. Mourad	P. Bouteiller	L. Salha	2021 – 2022
G. Bacquaert	S. Boulvard	Z. Chafia with J. Yvonnet LABEX MMCD	2023 – 2024
A. Gribonval	G. Blondet	G. d'Orio	started 2023

Others

2023 = **acting scientific director** of a European executive master on **Digital Twins for Infrastructures and Cities**

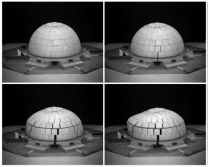
Outline

Outline

Introduction

Nothing takes place in the world whose meaning is not that of some maximum or minimum. (Leonhard Euler)

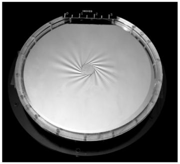
Collapse



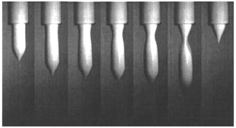
Masonry dome
[Zessin, 2015]

Landslides
[geologypage.com]

Thresholds

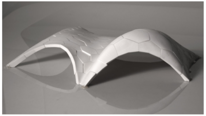


Membrane wrinkling
[shellbuckling.com]



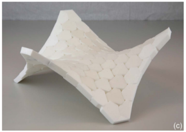
Mayonnaise drips
[Coussot et al, 2005]

Optimization



Vault [P. Block group, ETH]

RC beam [S. Maitenaz]



Topology interlocking assembly
[M. Pauly group, EPFL]

But also: finance, power networks, image processing, supply chain, machine learning, etc.

Convex variational problems

Differentiable case

$$\inf_{u\in V} J(u)$$

variational equality:
$D_u J(u,v) = 0 \quad \forall v \in V$

e.g. **potential energy**: $J(u) = \int_\Omega \psi(\nabla u)\,\mathrm{d}\Omega - \int_\Omega fu\,\mathrm{d}\Omega$

Convex variational problems

Cone-constrained case

$$\inf_{u\in V} \quad J(u)$$
$$\text{s.t.} \quad u \in \mathcal{K}$$

variational inequality:

$D_u J(u,v) \succeq_{\mathcal{K}^*} 0 \quad \forall v \in V$

Non-smooth case

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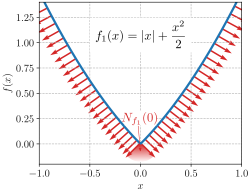
$D_u J(u,v) \ni 0 \quad \forall v \in V$



VI arise in presence of **inequality constraints** or **non-smooth** objective functions

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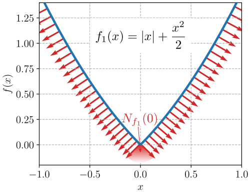
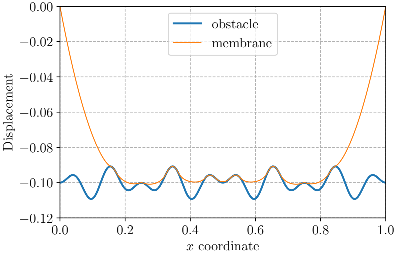


VI arise in presence of **inequality constraints** or **non-smooth** objective functions

Obstacle problem

$$\inf_{u\in V} \quad \int_\Omega \frac{1}{2}\|\nabla u\|_2^2 \,\mathrm{d}\Omega - \int_\Omega f u \,\mathrm{d}\Omega$$
$$\text{s.t.} \quad u \geq g \text{ on } \Omega$$

e.g. `PETSc` bound-constrained solvers



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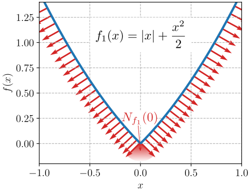

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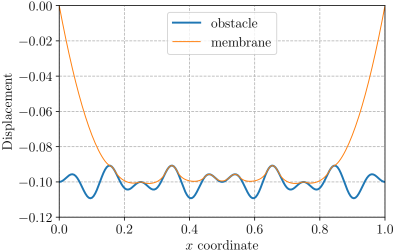

Motivation

non-smooth optimization problems entail **rich physics**, exhibit high **modeling expressiveness** and can now be **solved efficiently** using appropriate algorithms

Outline

Non-smooth optimization as conic programming

Linear programming

$$\begin{array}{rl} \min\limits_{\boldsymbol{x}} & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}$$

Non-smooth optimization as conic programming

Conic programming

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in \mathcal{K} \end{aligned}$$

Non-smooth optimization as conic programming

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where $\mathcal{K}$ is a product of elementary cones e.g.:

- positive orthants $\mathbb{R}^m_+$;
- Lorentz quadratic cones: $\mathcal{Q}_m = \{\mathsf{z} = (z_0, \bar{\mathsf{z}}) \in \mathbb{R}^+ \times \mathbb{R}^{m-1} \text{ s.t. } \|\bar{\mathsf{z}}\|_2 \leq z_0\}$
- semi-definite cones $\mathcal{S}^+_m$, the cone of positive semi-definite $m \times m$ symmetric matrices;
- power cones, exponential cones, etc.

Non-smooth optimization as conic programming

Conic programming

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in \mathcal{K} \end{aligned}$$

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The magic cone family [Juditsky & Nemirovski, 2021]

very large modelling power of **convex** functions and constraints

$$\begin{aligned} f(\boldsymbol{x}) = \min_{\boldsymbol{y}} \quad & \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T\boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{b}_l \leq \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{b}_u \\ & \boldsymbol{y} \in \mathcal{K}_1 \times \ldots \times \mathcal{K}_p \end{aligned} \qquad \text{(conic representation)}$$

Solvers

interior-point algorithms, very efficient and robust (20-30 iterations)

The dolfinx_optim package

```
prob = MosekProblem(domain, name="Obstacle problem")
u = prob.add_var(V, bc=bc, lx=g)

J = QuadraticTerm(ufl.grad(u), degree)
prob.add_convex_term(J)

prob.add_obj_func(-ufl.dot(f, u) * ufl.dx)

prob.optimize()
```

Obstacle problem

$$\begin{aligned} \inf_{u\in V} \quad & \int_\Omega \frac{1}{2}\|\nabla u\|_2^2 \, \mathrm{d}\Omega - \int_\Omega f u \, \mathrm{d}\Omega \\ \text{s.t.} \quad & u \geq g \text{ on } \Omega \end{aligned}$$

FEniCS PROJECT

$$\min_{\boldsymbol{u}\in V} \int_\Omega F(\boldsymbol{u}) \, \mathrm{d}\Omega$$

FE discretization $\Longrightarrow$ canonicalization

MOSEK

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in \mathcal{K} \end{aligned}$$

- **Domain-Specific Language** based on UFL for convex functions and their composition
- `Mosek` interior-point solver
- pre-defined **convex primitives**
  - `AbsValue`, `LinearTerm`, `QuadraticTerm`, `QuadOverLin`, etc.
  - vectors: `L1Norm`, `L2Norm`, `LinfNorm`, `LpNorm`, etc.
  - matrices: `SpectralNorm`, `NuclearNorm`, `FroebeniusNorm`, `LambdaMax`, etc.
- **composability** through convex-preserving transformations

Transformations

Convexity-preserving operations:

- sum $f_1(\boldsymbol{x}) + f_2(\boldsymbol{x})$
- supremum $\sup\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}$
- partial minimization
- Legendre-Fenchel transform

$$f^*(\boldsymbol{s}) = \sup_{x} \boldsymbol{s}^\mathsf{T}\boldsymbol{x} - f(\boldsymbol{x})$$

- inf-convolution

$$\begin{aligned}(f\Box g)(\boldsymbol{x}) = \inf_{\boldsymbol{x_1},\boldsymbol{x_2}} &\quad f(\boldsymbol{x}_1) + g(\boldsymbol{x}_2)\\ \text{s.t.} &\quad \boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2\end{aligned}$$

- perspective
- ...

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e.g. **perspective** function
$\text{persp}_f(t, \boldsymbol{x}) = tf(\boldsymbol{x}/t)$

$$\begin{aligned} tf(\boldsymbol{x}/t) = \min_{\boldsymbol{y}} \quad & \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{d}^T t\boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{b}_l \leq \boldsymbol{A}\boldsymbol{x}/t + \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{b}_u \\ & \boldsymbol{y} \in \mathcal{K}_1 \times \ldots \times \mathcal{K}_p \end{aligned}$$

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Data transformation

$$\boldsymbol{x} \mapsto \begin{Bmatrix} t \\ \boldsymbol{x} \end{Bmatrix} \qquad \boldsymbol{c} \mapsto \begin{Bmatrix} 0 \\ \boldsymbol{c} \end{Bmatrix}$$

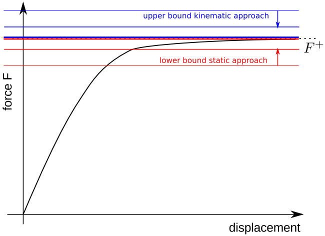
$$\boldsymbol{A} \mapsto \begin{bmatrix} -\boldsymbol{b}_l & \boldsymbol{A} \\ -\boldsymbol{b}_u & \boldsymbol{A} \end{bmatrix} \qquad \boldsymbol{B} \mapsto \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{B} \end{bmatrix}$$

$$\boldsymbol{b}_l \mapsto \begin{Bmatrix} 0 \\ \text{None} \end{Bmatrix} \qquad \boldsymbol{b}_u \mapsto \begin{Bmatrix} \text{None} \\ 0 \end{Bmatrix}$$

Outline

Limit analysis [Hill, Drucker, 1950] & Yield design [Salençon, 1983]

Goal: find the maximum collapse load $F^+ = \lambda^+ F$ that a structure can sustain under a convex plasticity domain G



Static approach = **limit load maximization**:

$$
\begin{aligned}
\lambda^+ = \max_{\lambda,\boldsymbol{\sigma}} \quad & \lambda \\
\text{s.t.} \quad & \boldsymbol{\sigma} \in \mathcal{S}_{\text{ad}}(\lambda) \quad \text{(equilibrium)} \\
& \boldsymbol{\sigma}(\boldsymbol{x}) \in G(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega
\end{aligned}
$$

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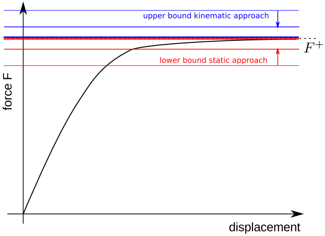

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Dual kinematic approach = **plastic dissipation minimization**:

$$\begin{aligned} \lambda^+ = \min_{\boldsymbol{u}\in\mathcal{U}_{\text{ad}}} \quad & \int_\Omega \pi_G(\boldsymbol{\varepsilon})\, \text{d}\Omega \\ \text{s.t.} \quad & \int_\Omega \boldsymbol{F}\cdot\boldsymbol{u}\, \text{d}\Omega = 1 \end{aligned}$$

$$\pi_G(\boldsymbol{\varepsilon}) = \sup_{\boldsymbol{\sigma}\in G} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}$$

exclusively non-smooth !

conic programming perfectly suited for that, impossible with standard Newton approach

Cohesive frictional soils

3D **Mohr-Coulomb** plasticity criterion

$$\boldsymbol{\sigma} \in G \Leftrightarrow \sigma_I - a\sigma_{III} \leq \frac{2c\cos\phi}{1+\sin\phi}$$

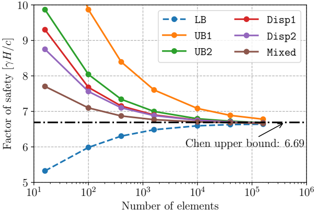
3D **Mohr-Coulomb** support function

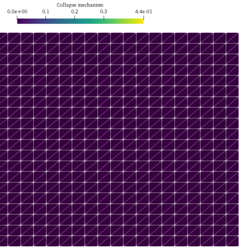
$$\pi_G(\boldsymbol{\varepsilon}) = \begin{cases} c\,\mathrm{cotan}\,\phi\,\mathrm{tr}\,\boldsymbol{\varepsilon} & \text{if } \mathrm{tr}(\boldsymbol{\varepsilon}) \geq \sin\phi \sum_I |\varepsilon_I| \\ +\infty & \text{otherwise} \end{cases}$$

Conic representation (SDP) of $\pi_G(\boldsymbol{\varepsilon})$

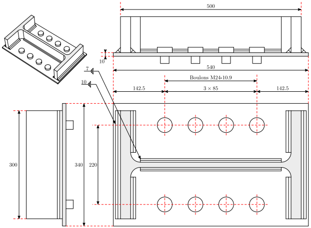
$$\begin{aligned} \pi_G(\boldsymbol{\varepsilon}) = \min_{\boldsymbol{Y}_1,\boldsymbol{Y}_2} \quad & \frac{2c\cos\phi}{1+\sin\phi}\mathrm{tr}(\boldsymbol{Y}_1) \\ \text{s.t.} \quad & \boldsymbol{\varepsilon} = \boldsymbol{Y}_1 - \boldsymbol{Y}_2 \\ & a\,\mathrm{tr}(\boldsymbol{Y}_1) = \mathrm{tr}(\boldsymbol{Y}_2) \\ & \boldsymbol{Y}_1 \succeq 0, \boldsymbol{Y}_2 \succeq 0 \end{aligned}$$

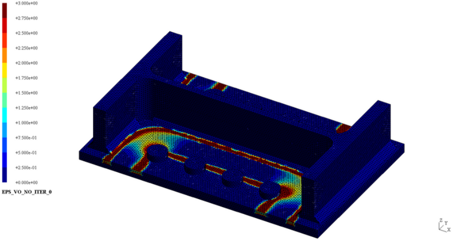
```
class MohrCoulomb(ConvexTerm):
    """SDP implementation of Mohr-Coulomb support function."""
    def conic_repr(self, X):
        Y1 = self.add_var((3,3), cone=SDP(3))
        Y2 = self.add_var((3,3), cone=SDP(3))
        a = (1 - ufl.sin(phi)) / (1 + ufl.sin(phi))
        self.add_eq_constraint(X - to_vect(Y1) + to_vect(Y2))
        self.add_eq_constraint(ufl.tr(Y2) - a * ufl.tr(Y1))
        self.add_linear_term(2 * c * ufl.cos(phi) / (1 +
       ufl.sin(phi)) * ufl.tr(Y1))
```



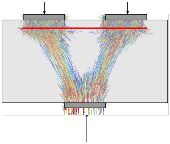


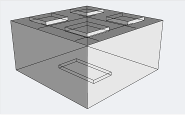
Bolted column base plate [C. El Boustani]

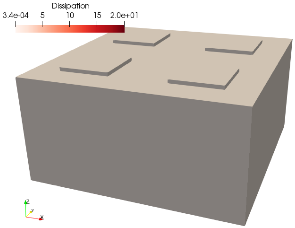




Reinforced concrete bridge pier cap [H. Vincent]



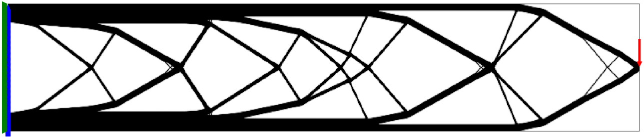




Outline

Topology optimization : elastic setting

Find $\Omega \subseteq \mathcal{D}$ minimizing the **elastic compliance** at fixed volume [Allaire, 2002]:



[TopOpt in Python, DTU]

$$
\begin{aligned}
\min_{\Omega, \boldsymbol{u}} \quad & \int_{\partial\Omega} \boldsymbol{T} \cdot \boldsymbol{u} \, \mathrm{dS} \\
\text{s.t.} \quad & \boldsymbol{\sigma} = \mathbb{C} : \nabla \boldsymbol{u} \text{ in } \Omega \\
& \operatorname{div} \boldsymbol{\sigma} = 0 \quad \text{in } \Omega \\
& \boldsymbol{\sigma} \boldsymbol{n} = \boldsymbol{T} \quad \text{on } \partial\Omega_N \\
& |\Omega| \leq \eta |\mathcal{D}|
\end{aligned}
$$

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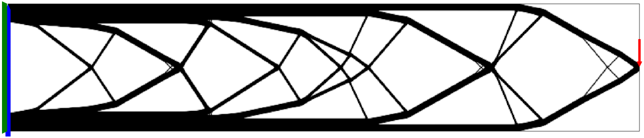

[TopOpt in Python, DTU]

Density-based formulation

$$\begin{aligned}
\min_{\rho,\boldsymbol{u}} \quad & \int_{\partial\mathcal{D}} \boldsymbol{T}\cdot\boldsymbol{u}\,\mathrm{dS} \\
\text{s.t.} \quad & \boldsymbol{\sigma} = \mathbb{C}(\rho) : \nabla\boldsymbol{u} \text{ in } \mathcal{D} \\
& \operatorname{div}\boldsymbol{\sigma} = 0 \quad \text{in } \mathcal{D} \\
& \boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{T} \quad \text{on } \partial\mathcal{D}_N \\
& \textstyle\int_{\mathcal{D}} \rho\,\mathrm{d}\Omega \leq \eta|\mathcal{D}| \\
& 0 \leq \rho(\boldsymbol{x}) \leq 1
\end{aligned}$$

$\Rightarrow$ **non-convex** problem, iterative procedure
e.g. SIMP method [Bendsoe and Kikuchi, 1988] : $\mathbb{C}(\rho) = \rho^p\mathbb{C}_0$ with $p > 1$

Maximizing the limit load [Mourad et al., 2021]

Objective : Find $\Omega \subseteq \mathcal{D}$ with **maximum limit load** for a given volume level η:

Proposed formulation:

$$\begin{array}{rll} \lambda^+(\eta) = \displaystyle\max_{\lambda,\boldsymbol{\sigma},\Omega} & \lambda & \\ \text{s.t.} & \operatorname{div}\boldsymbol{\sigma} = 0 & \text{in } \Omega \\ & \boldsymbol{\sigma}\boldsymbol{n} = \lambda \boldsymbol{T} & \text{in } \partial\Omega_N \\ & \boldsymbol{\sigma} \in G & \text{in } \Omega \\ & |\Omega| \leq \eta|\mathcal{D}| & \end{array}$$

Maximizing the limit load [Mourad et al., 2021]

Objective : Find $\Omega \subseteq \mathcal{D}$ with **maximum limit load** for a given volume level η:

extension by $\boldsymbol{\sigma} = 0$ outside Ω

$$
\begin{aligned}
\lambda^+(\eta) = \max_{\lambda,\boldsymbol{\sigma},\rho} \quad & \lambda \\
\text{s.t.} \quad & \operatorname{div} \boldsymbol{\sigma} = 0 && \text{in } \mathcal{D} \\
& \boldsymbol{\sigma} \boldsymbol{n} = \lambda \boldsymbol{T} && \text{in } \partial\mathcal{D}_N \\
& \boldsymbol{\sigma} \in \rho G && \text{in } \mathcal{D} \\
& \textstyle\int_{\mathcal{D}} \rho \, \mathrm{d}\Omega \leq \eta |\mathcal{D}| \\
& \rho \in \{0; 1\}
\end{aligned}
$$

ρ being the characteristic function of Ω

Maximizing the limit load [Mourad et al., 2021]

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problem **convexification** (LOAD-MAX)

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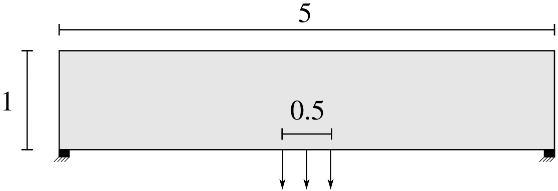
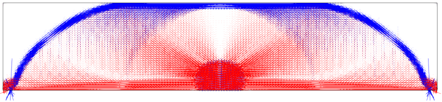
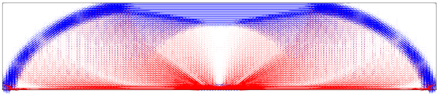
- **convex problem**, akin to a limit analysis problem with an additional scalar variable ρ
- related to a **volume minimization** problem [Kammoun, 2014; Herfelt et al., 2019]
- **penalization procedure** $\rho^p \approx \rho_n^{p_n} + p_n\rho_n^{p_n-1}\rho \quad \Rightarrow$ **black-and-white** design
- **slope limitation** [Petersson & Sigmund, 1998] $\|\nabla\rho\| \leq 1/\ell_0$ for **mesh independency**

Material with asymmetric strengths





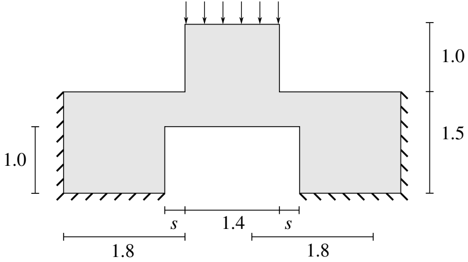
(a) $f_c/f_t = 10$

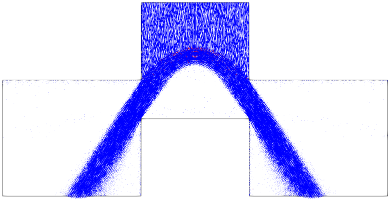


(b) $f_c/f_t = 0.1$

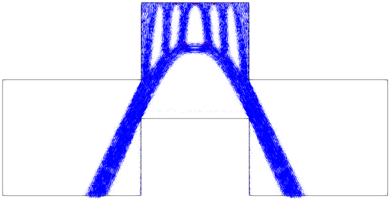
Principal stresses (compression/traction)

Material without tensile strength



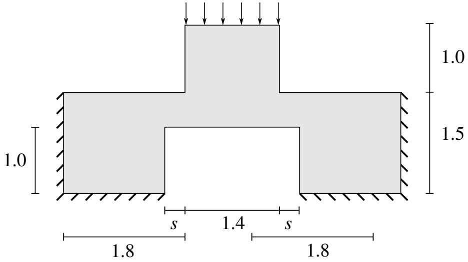


(a) Before penalization

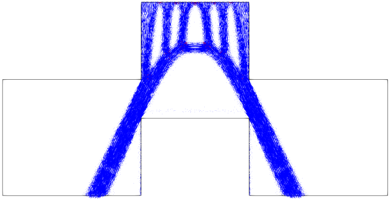


(b) After penalization

Material without tensile strength



(a) The Passion Façade



(b) After penalization

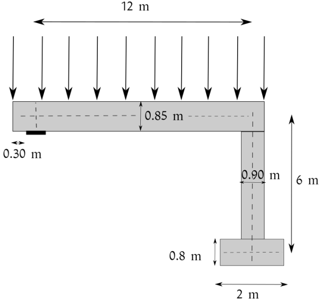
Extension to two materials

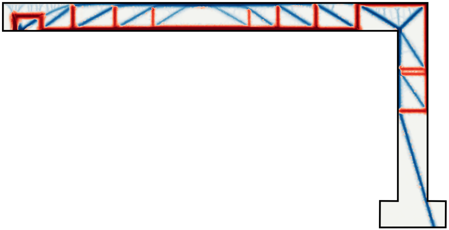
we want to optimize independently two phases (+ void) e.g. **steel** and **concrete**, **tension** and **compression**

Strength condition

$$\begin{cases} \boldsymbol{\sigma} = \boldsymbol{\sigma}^1 + \boldsymbol{\sigma}^2 \\ \boldsymbol{\sigma}^1 \in \rho_1 G^1 \\ \boldsymbol{\sigma}^2 \in \rho_2 G^2 \end{cases} \quad \text{with } \rho_1 + \rho_2 \leq 1$$

G^1 possibly **anisotropic**



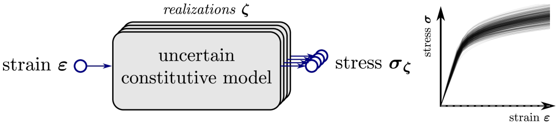


Outline

Objectives

Material constitutive law: $\boxed{\boldsymbol{\sigma} = \mathsf{F}(\boldsymbol{\varepsilon})}$

uncertain material properties $\Rightarrow$ need for an **effective behavior** [Bleyer, JMPS, 2024]



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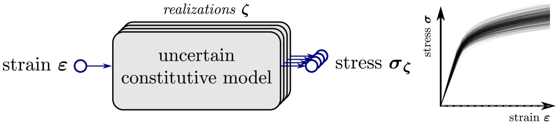

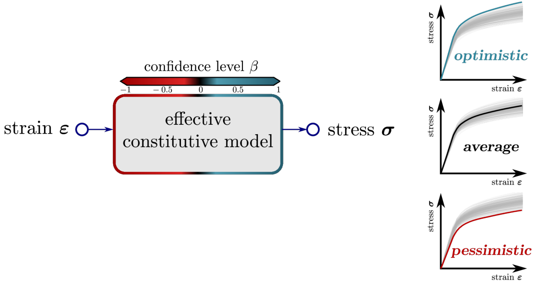

must account for **history-dependent** behaviors and **thermodynamic consistency**

Evolution equations of a standard material

Incremental potential

After time discretization, evolution equations of **Generalized Standard Materials** are obtained from the **incremental potential** [Ortiz & Stainier, Mielke, etc.]

$$\psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) + \Delta t \phi\left(\frac{\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_n}{\Delta t}, \frac{\boldsymbol{\alpha}-\boldsymbol{\alpha}_n}{\Delta t}\right)$$

Simplifying assumptions: $\boldsymbol{\varepsilon}$ is non-dissipative, ϕ is 1-homogeneous, single-step $\boldsymbol{\alpha}_n = 0$

$$j(\boldsymbol{\varepsilon}) := \inf_{\alpha} \psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) + \phi(\boldsymbol{\alpha})$$

$$\Rightarrow \boldsymbol{\sigma} \in \partial_{\boldsymbol{\varepsilon}} j$$

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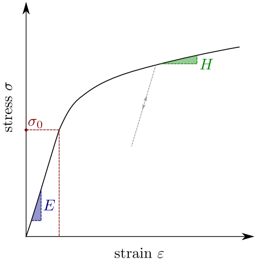
$$\Rightarrow \boldsymbol{\sigma} \in \partial_{\boldsymbol{\varepsilon}} j$$

Example: 1D linear elasticity + isotropic power-law hardening

$$\begin{aligned}\psi(\varepsilon,\alpha) &= \psi_{\text{el}}(\varepsilon-\alpha) + \psi_{\text{h}}(\alpha) \\ &= \frac{1}{2}E(\varepsilon-\alpha)^2 + \frac{1}{m}H\alpha^m \\ \phi(\dot{\alpha}) &= \sigma_0|\dot{\alpha}|\end{aligned}$$



Uncertain elastoplastic case

Now j depends upon **stochastic parameters** ζ with known probability distribution
Goal: formulate an **effective potential** to describe the effective behavior

$$\boxed{j^{\text{eff}}(\varepsilon) = \mathcal{R}\left[j(\varepsilon;\zeta)\right]}$$

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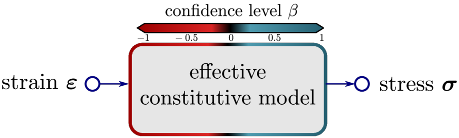

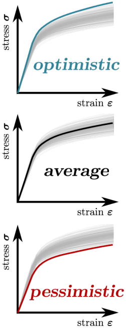

Risk-averse measures

Risk-measure $\mathcal{R}$: assume X be a **random cost**: $\mathbb{E}[X] =$ OK whereas $\mathcal{R}[X] =$ BAD e.g.:

- the safety margin $\mathcal{R}[X] = \mathbb{E}[X] + k\,\text{std}[X], \quad \text{for } k > 0$
- the worst-case value: $\mathcal{R}[X] = \sup X$
- the *Value-at-Risk* (VaR) for a level $\beta \in [0; 1]$ (or the β-quantile):

$$\mathcal{R}[X] = \text{VaR}_\beta(X) = \inf\{Z \text{ s.t. } \mathbb{P}[Z \geq X] \geq \beta\}$$

- and many more...

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$\Rightarrow$ **coherent risk measures** [Artzner, 1999]

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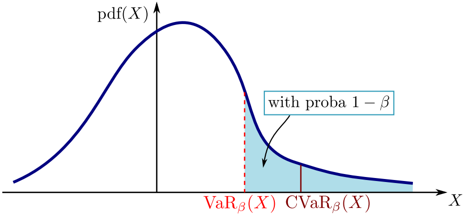
we look for **good mathematical properties** such as **convexity**, **monotonicity**, **homogeneity** $\Rightarrow$ **coherent risk measures** [Artzner, 1999]

- **safety margin** and **VaR** are **not coherent**
- **worst-case value** is coherent but **too conservative**
- **expected value** is coherent but **risk-neutral**

Conditional Value-at-Risk (CVaR)

The **Conditional Value-at-Risk** (CVaR) is a **coherent risk measure** [Rockafellar, 2000]

$$\mathrm{CVaR}_\beta(X) = \mathbb{E}[X \text{ s.t. } X \geq \mathrm{VaR}_\beta(X)]$$



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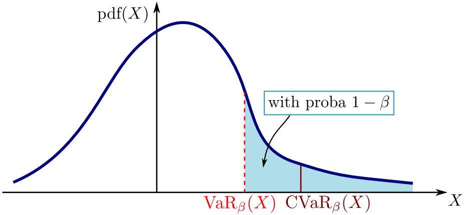

Key result: **convex optimization characterization**

$$\mathrm{CVaR}_\beta(X) = \inf_\lambda \lambda + \frac{1}{1-\beta}\mathbb{E}\left[\langle X - \lambda\rangle_+\right]$$

Extends to **random convex functions**:

$$\boxed{\mathrm{CVaR}_\beta(f)(\boldsymbol{x}) = \inf_\lambda \lambda + \frac{1}{1-\beta}\mathbb{E}\left[\langle f(\boldsymbol{x};\zeta) - \lambda\rangle_+\right]} \textbf{ is convex}$$

Examples

1D elasticity: $j(\varepsilon_\zeta;\zeta) = \dfrac{1}{2}E_\zeta\varepsilon_\zeta^2$

$$j^{\text{eff}}(\varepsilon) = \text{CVaR}_\beta\left(\psi\right)(\varepsilon) = \frac{1}{2}\text{CVaR}_\beta\left(E\right)\varepsilon^2$$

replaces uncertain Young modulus E_ζ with an **optimistic estimate** $\text{CVaR}_\beta\left(E\right)$

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General case

take **CVaR** on **free-energy** and **dissipation potential** + $\boldsymbol{\varepsilon}$ first-stage

$$j^{\text{eff}}(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{\alpha}_\zeta} \text{CVaR}_\beta\left(\psi(\boldsymbol{\varepsilon},\boldsymbol{\alpha}_\zeta;\zeta)\right) + \text{CVaR}_\beta\left(\phi(\boldsymbol{\alpha}_\zeta;\zeta)\right)$$

- if $\beta = 0$, $\text{CVaR} = \mathbb{E}$ and we recover **average formulation**
- results in **optimistic** stiffness, strength and hardening

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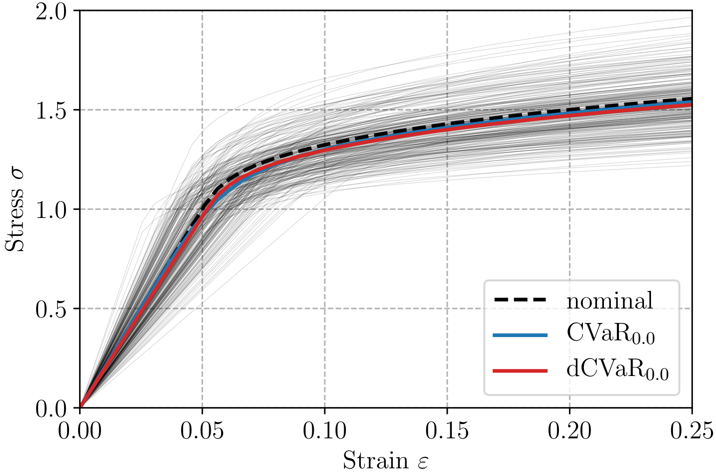
Pessimistic estimates

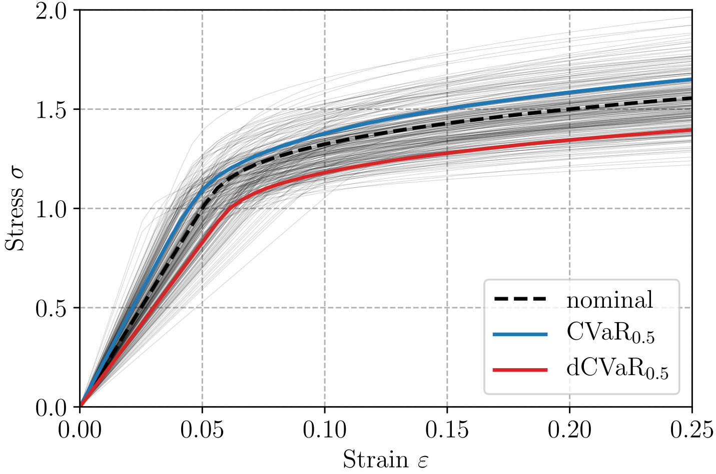
need to define an **original dual** CVaR

Numerical results



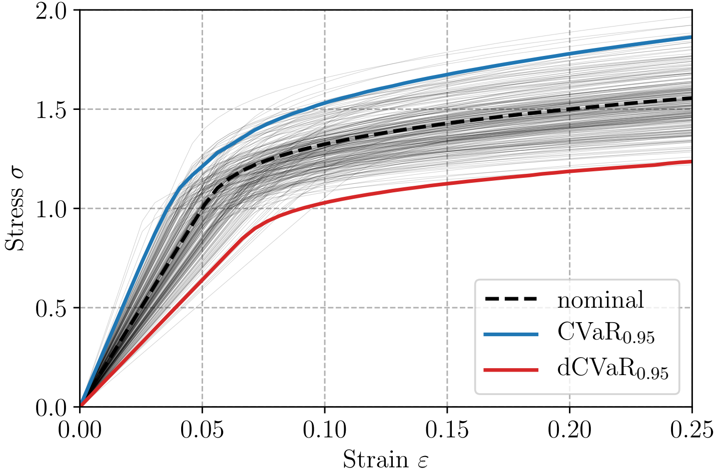
risk-neutral case $\beta = 0$

Numerical results



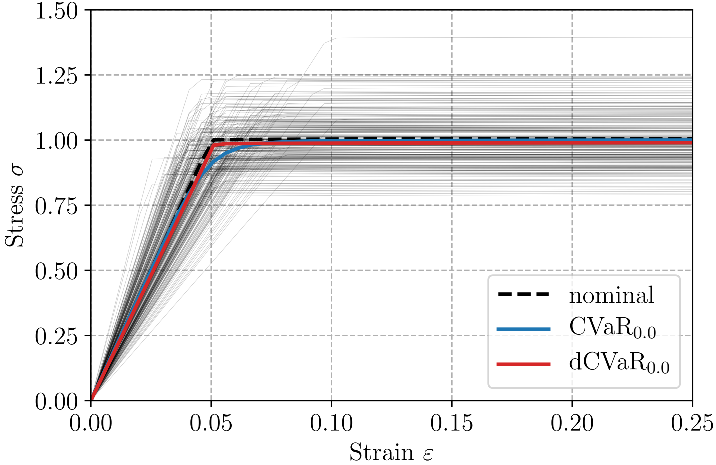
moderate risk-aversion $\beta = 0.5$

Numerical results



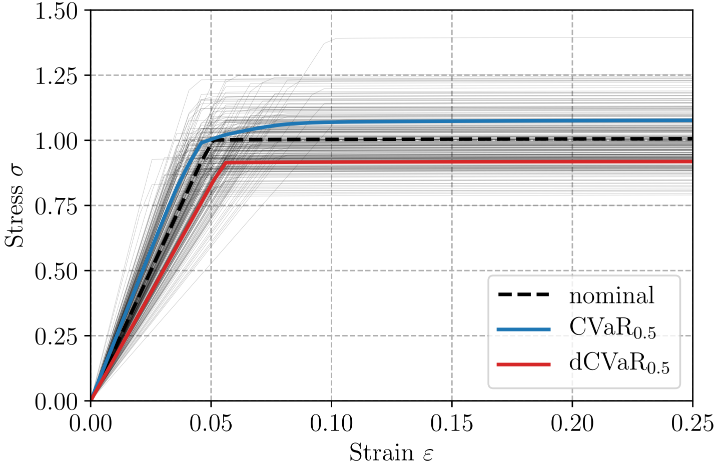
strong risk-aversion $\beta = 0.95$

Numerical results



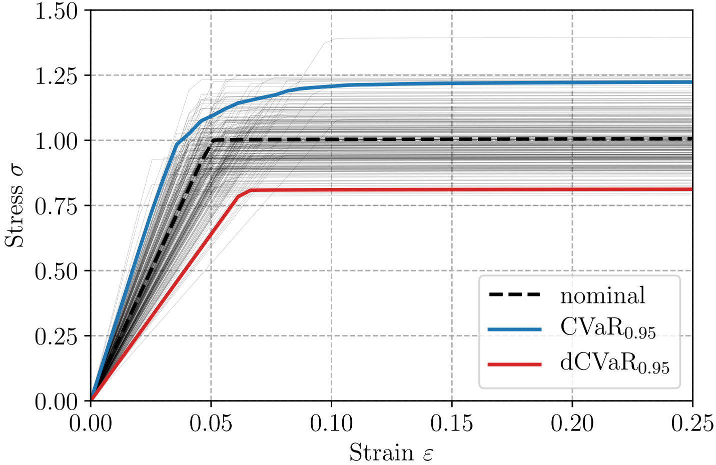
risk-neutral case $\beta = 0$

Numerical results



moderate risk-aversion $\beta = 0.5$

Numerical results



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Risk-averse stochastic programming formulation at the structure scale

Work directly on the **global** free-energy and dissipation potentials:

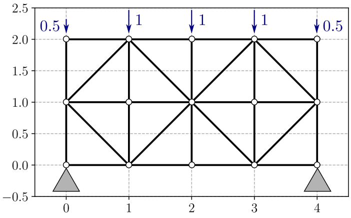
Optimistic formulation:

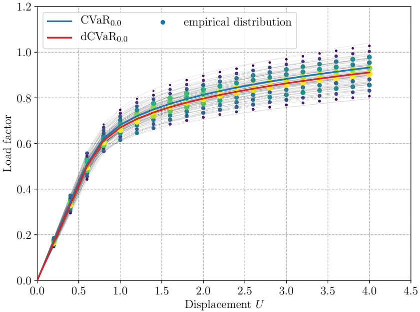
$$\boldsymbol{u}_{n+1}, \alpha_{\boldsymbol{\zeta},n+1} = \underset{\boldsymbol{u}\in\mathcal{U}_{\text{ad}},\boldsymbol{\alpha}_{\boldsymbol{\zeta}}}{\arg\inf}\ \text{CVaR}_\beta(\Psi)(\boldsymbol{\varepsilon},\boldsymbol{\alpha}_{\boldsymbol{\zeta}}) + \text{CVaR}_\beta(\Phi)(\boldsymbol{\alpha}_{\boldsymbol{\zeta}}) - \langle \boldsymbol{F}, \boldsymbol{u}\rangle$$

Pessimistic formulation:

$$\boldsymbol{u}_{n+1}, \alpha_{\boldsymbol{\zeta},n+1} = \underset{\boldsymbol{u}\in\mathcal{U}_{\text{ad}},\boldsymbol{\alpha}_{\boldsymbol{\zeta}}}{\arg\inf}\ \text{dCVaR}_\beta(\Psi)(\boldsymbol{\varepsilon},\boldsymbol{\alpha}_{\boldsymbol{\zeta}}) + \text{dCVaR}_\beta(\Phi)(\boldsymbol{\alpha}_{\boldsymbol{\zeta}}) - \langle \boldsymbol{F}, \boldsymbol{u}\rangle$$

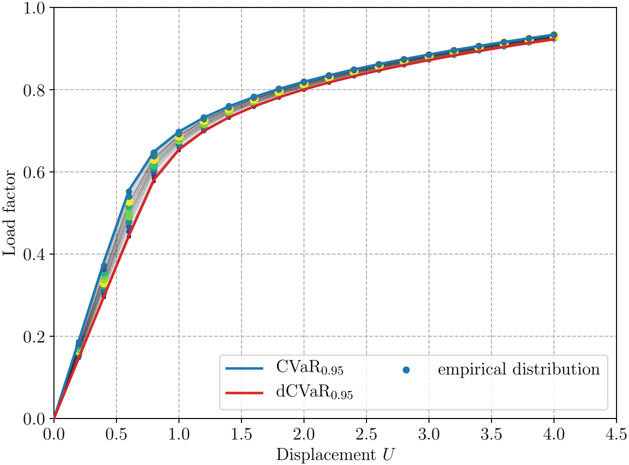
Truss structure with members obeying stochastic elastoplastic hardening behaviour





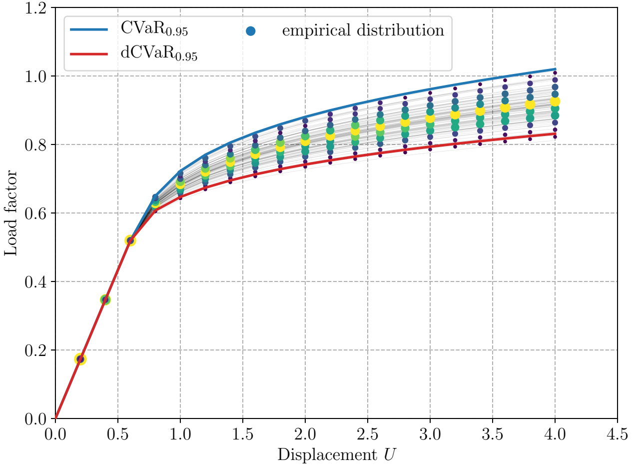
Risk-averse case

Risk-averse case: uncertainty on **Young modulus only**



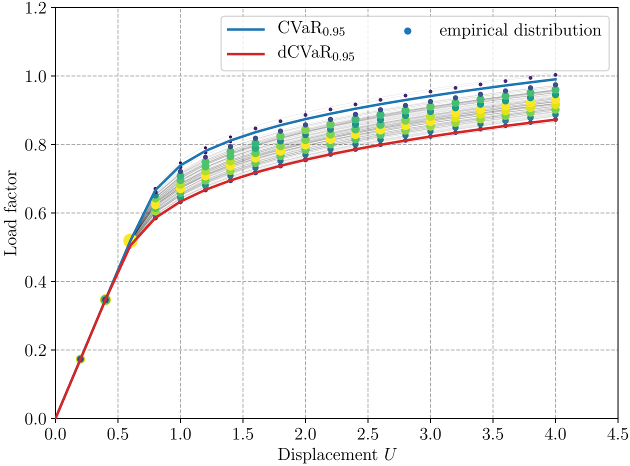
Risk-averse case

Risk-averse case: uncertainty on **hardening modulus only**



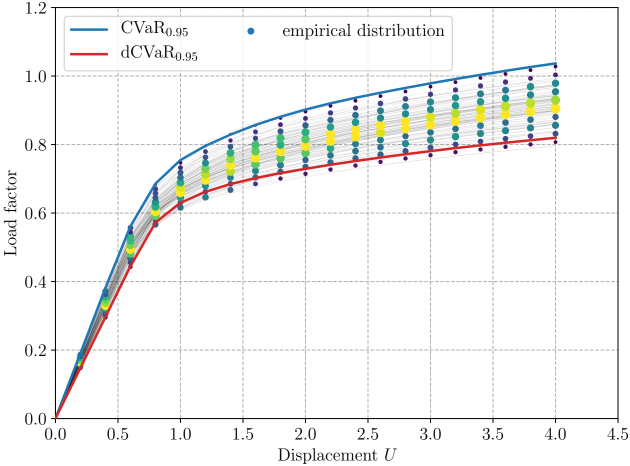
Risk-averse case

Risk-averse case: uncertainty on **yield stress only**

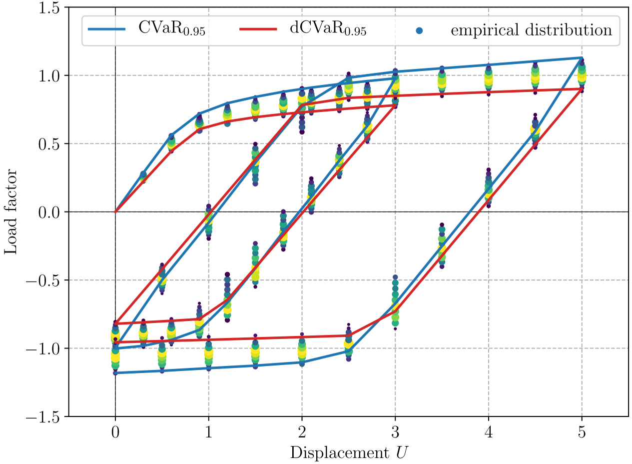


Risk-averse case

Risk-averse case: combined uncertainty



Cyclic loading



Outline

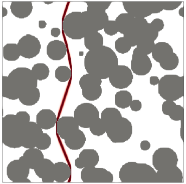
General conclusions

Conic optimization has **many things to offer**:

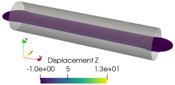
- **rich physics**: non-smooth, yielding behaviors
- **modeling expressiveness**: viscoplastic fluids, minimal cracks, membranes and shells, image processing, optimal transport
- **numerical robustness**: efficiency wrt Newton-Raphson, variational integration

Nonlinear membranes

Crack plane orientation: 0°



Minimal crack surface



Funicular form-finding

Challenges & opportunities

- integration in industrial software
- hybrid methods (Newton or first-order), warm-start
- preconditioning

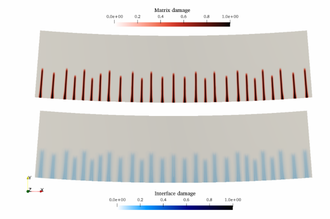
Research perspectives

Optimization under uncertainty

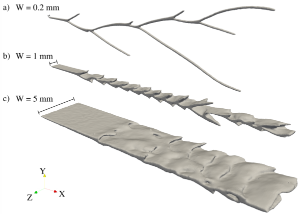
- fertile links to be drawn with **robust optimization** and **stochastic programming**
- **computational cost reduction** for adjustable formulations
- inclusion with **robust topology optimization**

Regularization of softening behaviors

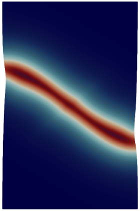
- **alternate minimization** or **fixed-point** iterations of convex sub-problems
- **damage gradient/phase-field** models for heterogeneous materials
- softening plasticity: **dissipation-based regularization** [Bacquaert et al., JMPS, 2024]



fiber-reinforced fracture



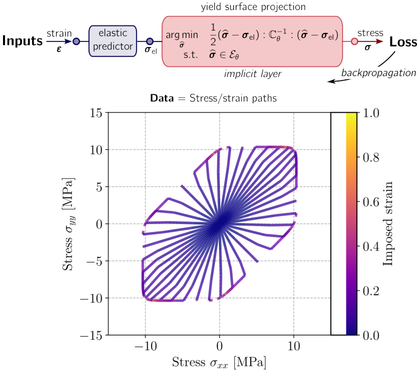
dynamic crack branching

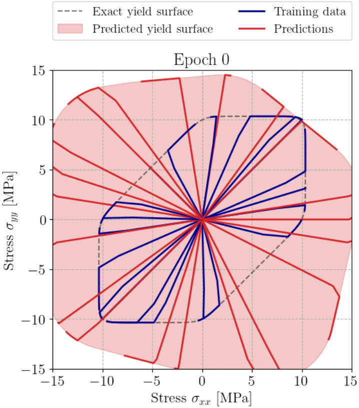


softening plasticity

Towards a Machine Learning era in computational mechanics ?

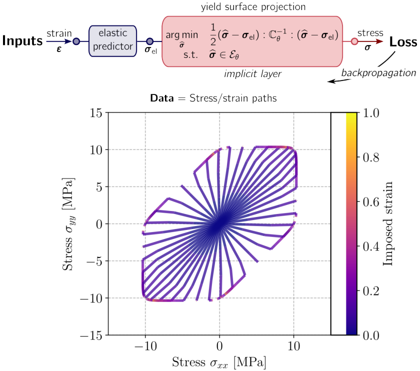
Learning plastic yield surfaces using implicit layers

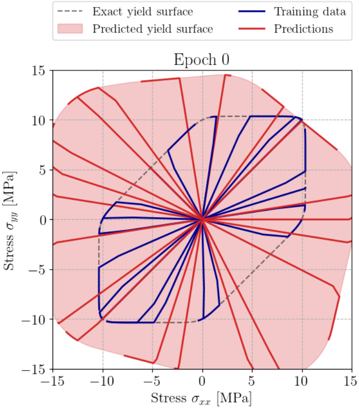




Towards a Machine Learning era in computational mechanics ?

Learning plastic yield surfaces using **implicit layers**





Thank you for your attention !